HARDER-NARASIMHAN FILTRATION

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Contents

1. Introduction

This note is a summary of the theory of Harder-Narasimhan filtration and aims to be accessible. As such, we will not record the proofs as they can get quite technical and cloud the big picture. However, when possible we will try to sketch the key ideas contained in the proof. The main references are K. Behrend's PhD thesis [Beh91] and a published paper extracted from his thesis [Beh95], a more recent alternative approach by S. Schieder [Sch15], and some sections of [GL19] which contains a concise review of this theory. The language in Behrend's writings is somewhat different from the other two more modern accounts, and we will also try to bridge the two.

Throughout this note, k is a field and X is a smooth, geometrically connected and projective curve over k. Let G be a connected reductive group scheme over X, in other words, $G \to X$ is a smooth affine group scheme whose geometric fibers are reductive and connected.

2. Motivations

2.1. Before we introduce the notion of stability of arbitrary reductive groups, let us look at a more classical situation. Consider a vector bundle $\mathcal{E}$ of rank n on X. Then we can define its degree as the degree of its determinant line bundle $\wedge^n\mathcal{E}$

$$\deg \mathcal{E} \coloneqq \deg \wedge^n \mathcal{E}.$$

We then define the notion of *slope* introduced by D. Mumford:

$$\mu(\mathcal{E}) \coloneqq \frac{\deg \mathcal{E}}{\operatorname{rk} \mathcal{E}} = \frac{\deg \mathcal{E}}{n}.$$

More generally, for any coherent sheaf on X, one can define its slope to be the slope of its torsion-free quotient (as X is a smooth curve, a torsion-free coherent sheaf is locally-free). The bundle $\mathcal{E}$ is called *semi-stable* if for any subsheaf $\mathcal{F} \subset \mathcal{E}$ we have $\mu(\mathcal{F}) \le \mu(\mathcal{E})$, and *stable* if the inequality is strict for any proper subsheaf $\mathcal{F}$.

2.2. If you are like me, this notion seems to come out of nowhere and lacks geometric intuition. Nevertheless, it is very useful in moduli problems, and it seems to first arise out of technical necessity when studying such problems. For example, in our situation, we want to consider the moduli stack Bun_G, and when $G = \mathrm{GL}_n$ it classifies all vector bundles of rank n on X, hence $\mathcal{E}$ can be seen as a k-point of Bun_G. However, Bun_G is not very nice in that it is not a moduli *space* and cannot be approximated by such (as a whole). The problem comes from automorphism groups, which we illustrate below using two basic examples.

Example 2.3. If $G = \mathrm{GL}_1 = \mathbb{G}_{\mathrm{m}}$, then Bun_G is the same as the Picard *stack* $\mathcal{P}\mathrm{ic}\, X$. Although it is an Artin stack, since any line bundle on X has automorphism group $\mathbb{G}_{\mathrm{m}}$, it nonetheless admits a coarse space the Picard *scheme* $\mathrm{Pic}\, X$ by simply taking isomorphism classes of $\mathcal{P}\mathrm{ic}\, X$ (which turns out to be a representable sheaf). The natural morphism $\mathcal{P}\mathrm{ic}\, X \to \mathrm{Pic}\, X$ is a $\mathbb{G}_{\mathrm{m}}$-gerbe which can be trivialized by choosing a universal line bundle on $\mathrm{Pic}\, X$. In this case, everything seems good because for any line bundle, its automorphism group is always equal to $\mathbb{G}_{\mathrm{m}}$, thus we are able to eliminiate them "uniformly" to reduce a stack to a space.

Example 2.4. When $G = \mathrm{GL}_2$, then Bun_G no longer admits a coarse moduli space. Indeed, let D be an effective Cartier divisor on X such that $\mathcal{O}_X(D)$ is generated by global sections. For each $i \geq 0$, let $\mathcal{E}_i = \mathcal{O}_X(iD) \oplus \mathcal{O}_X(-iD)$, then it can be seen as a trivial extension of $\mathcal{O}_X(iD)$ by $\mathcal{O}_X(-iD)$, i.e., an element of cohomology group

$$T_i = \mathrm{Ext}^1_X(\mathcal{O}_X(iD), \mathcal{O}_X(-iD)) = \mathrm{H}^1(X, \mathcal{O}_X(-2iD)),$$

viewed as a k-affine space classifying all such extensions. In particular, we have a map $T_i \to \mathrm{Bun}_G$. Since $\mathcal{O}_X(D)$ is generated by global sections, we see that each $\mathcal{E}_i$ is an extension of $\mathcal{O}_X((i+1)D)$ by $\mathcal{O}_X(-(i+1)D)$. Using semicontinuity result of flat coherent sheaves, we see that the same is true for every bundle in T_i. In other words, the map $T_i \to \mathrm{Bun}_G$ factors through $T_{i+1} \to \mathrm{Bun}_G$. Clearly, different points in T_i gives non-isomorphic vector bundles, therefore if a coarse moduli space M exists for Bun_G, then any connected component of M contains affine spaces of arbitrary dimensions, which is impossible because Bun_G is locally of finite type. On the other hand, such is possible for Bun_G itself because the automorphism groups of $\mathcal{E}_i$ have unbounded dimensions, which "negatively compensate" the local dimension to be constant.

2.5. The above example with GL_2 contains an additional hint: we may be able to stratify Bun_G such that each stratum can be approximated. This is exactly what Harder-Narasimhan filtration is about and Mumford's slope turns out to be a special case under some reformulation.

3. Forms of Reductive Groups

In this section we record some well-known facts about automorphisms and forms of reductive groups. Readers familiar with these facts can safely skip this section and only occasionally come back for notations. Most of the results can be found, say, in SGA3, but we do try to explain things in an elementary way if we can.

3.1. First we consider G a split reductive group over $\bar{k}$. For any split maximal torus T, there is an associated root datum which depends on the choice of the torus. However, if one also chooses a Borel subgroup B containing T, then the induced *based* root datum can be made canonical as follows: any other choice of pair (T', B') is conjugate to (T, B) by some element $g \in G$, and any choice of g induces the same isomorphism from T to T'. Hence the induced isomorphism on based root data is canonical. Taking the inductive limit of all choices of pairs (T, B), we obtain the canonical based root datum

$$(\mathbb{X}, \Delta, \check{\mathbb{X}}, \check{\Delta}),$$

where $\mathbb{X}$ is the character lattice and Δ is the set of simple roots, and $\check{\mathbb{X}}$ and $\check{\Delta}$ are the respective dual notions. Alternatively, we may identify $\mathbb{X}$ with the character lattice of quotient torus $B/U_B \simeq T$, and this identification is canonical because the normalizer of B in G is B itself.

3.2. We continue with G over $\bar{k}$. The group G acts on itself by conjugation, in other words, we have a natural map

$$G \longrightarrow \mathrm{Aut}(G)$$

whose kernel is the center Z_G. Let $G^{\mathrm{ad}} = G/Z_G$ be the adjoint group, then G^{ad} is naturally identified with the subgroup $\mathrm{Int}(G) \subset \mathrm{Aut}(G)$, called the inner automorphisms of G. It is clearly a normal subgroup, and the quotient $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$ is the group of outer automorphisms. In fact, we can make certain choices to split the exact sequence

$$1 \longrightarrow \mathrm{Int}(G) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1.$$

Indeed, we choose Borel pair (T, B) and for each simple root $\alpha \in \Delta$ we choose a non-zero root vector $x_\alpha \in U_\alpha$, where $U_\alpha \subset U_B$ is the one-parameter unipotent subgroup corresponding to α. The datum

$(T, B, \{x_\alpha\}_{\alpha\in\Delta})$ is then called a *pinning* or *épinglage* of G. If G is defined and split over k, these choices can be made over k.

Given any automorphism ϕ of G, it maps Borel B to another Borel, and since over $\bar{k}$ all Borel subgroups are conjugate, we may find $g \in G$ such that $g\phi(B)g^{-1} = B$. This means that $\mathrm{Aut}(G)$ is generated by G^{ad} together with the subgroup $\mathrm{Aut}(G, B) \subset \mathrm{Aut}(G)$ consisting of automorphisms sending B to B itself. Using similar argument, it is easy to see that the automorphisms fixing a pinning $(T, B, \{x_\alpha\}_{\alpha\in\Delta})$ has trivial intersection with G^{ad} and together they generate $\mathrm{Aut}(G)$. In other words, we have isomorphism

$$\mathrm{Aut}(G, T, B, \{x_\alpha\}_{\alpha\in\Delta}) \xrightarrow{\sim} \mathrm{Out}(G).$$

This identifies $\mathrm{Aut}(G)$ with a semi-direct product $\mathrm{Int}(G)\rtimes\mathrm{Out}(G)$. Finally, using the construction of Chevalley groups, one may show that given an abstract based root datum there exists a unique split reductive group with a pinning, up to a unique isomorphism. This implies we have another isomorphism

$$\mathrm{Aut}(G, T, B, \{x_\alpha\}_{\alpha\in\Delta}) \simeq \mathrm{Aut}(\mathbb{X}, \Delta, \check{\mathbb{X}}, \check{\Delta}).$$

3.3. It is known that any reductive group scheme over a non-empety base scheme (not necessarily a curve) becomes split after passing to an étale cover. It is mainly deduced from an existence result of maximal tori which we will not cover here.

Therefore one can glue together the canonical based root datum over X, and view $\mathbb{X}$ as a countable union of *finite* étale cover of X. Therefore we can always find a finite étale cover $X' \to X$, such that the canonical based root datum becomes split over X'. Consider a split group $\mathbf{G}$ with a pinning $(\mathbf{T}, \mathbf{B}, \{\mathbf{x}_\alpha\}_{\alpha\in\Delta})$ over X' corresponding to this based root datum. It is straightforward to see from our discussion above that the sheaf of group isomorphisms

$$\underline{\mathrm{Isom}}_{X'}(\mathbf{G}, G \times_X X')$$

is a torsor under constant group $\mathrm{Aut}(\mathbf{G})$, and the subsheaf of automorphisms fixing the canonial based root datum is a torsor E^{ad} under constant group $\mathbf{G}^{\mathrm{ad}}$. Moreover, we have isomorphism

$$\begin{aligned} E^{\mathrm{ad}} \times^{\mathbf{G}^{\mathrm{ad}}} \mathbf{G} &\xrightarrow{\sim} G \times_X X' \\ (\phi, g) &\longmapsto \phi(g). \end{aligned}$$

In other words, for any reductive group $G \to X$, one can find a (connected) *finite* étale cover $X' \to X$, over which G becomes an inner form. Here we do not need X to be a curve.

By further passing to some finite étale cover, we may assume E^{ad} is induced by a torsor E^{sc} of the simply-connected group $\mathbf{G}^{\mathrm{sc}}$. If X is a curve and k is either finite or algebraically closed, then E^{sc} is necessarily trivial after restricting to the generic point of X. Using the map $\mathbf{G}^{\mathrm{sc}} \to \mathbf{G}$, we may also use the $\mathbf{G}$-torsor E induced by E^{sc}. In addition, one can always assume the étale cover is actually Galois. Thus, we have the following result.

Proposition 3.4. *For any reductive group scheme G over a curve X, there exists a connected finite Galois étale cover $X' \to X$ such that $G \times_X X'$ is an inner form and its restriction to the generic point of X' is split.*

3.5. Finally, we can also descend the split group $\mathbf{G}$ from X' to X by letting $\mathrm{Out}(\mathbf{G})$ acts on $\mathbf{G}$ through the fixed pinning. We denote the result by $\mathbf{G}'$, and it is called a *quasi-split* form. It comes with a pair $(\mathbf{T}', \mathbf{B}')$ which is the descent of $(\mathbf{T}, \mathbf{B})$, and the only difference is that the maximal torus $\mathbf{T}'$ is no longer split. The set of root vectors x_α can be descended too but must be viewed as a finite étale cover of X. We will simply call this cover $\mathbf{x}'$, we call $(\mathbf{T}', \mathbf{B}', \mathbf{x}')$ a *pinning* of $\mathbf{G}'$. The subsheaf of $\underline{\mathrm{Isom}}_X(\mathbf{G}', G)$ fixing the common canonical based root datum is a torsor under $(\mathbf{G}')^{\mathrm{ad}}$. We leave the details to the reader.

4. Parabolic Subgroups

In this section, we summarize some facts about parabolic subgroups of G. We retain all notations from the previous section, and readers familiar with the results may safely skip this section and only come back for references.

4.1. Again, we start with the absolute case where G is a reductive group over $\bar{k}$. In this case, a parabolic subgroup P of G is such that the geometric quotient G/P is representable by a projective $\bar{k}$-variety. It is well-known that P contains some Borel subgroup B, and given a maximal torus T in G, any P containing T may be obtained by specifying a cocharacter $\check{\lambda} \in \check{\mathbb{X}}(T)$, such that $P = P(\check{\lambda})$ is generated by T and the one-parameter unipotent subgroups corresponding to roots α such that $\langle \alpha, \check{\lambda} \rangle \geq 0$. If we also fix a Borel B containing T, then any P containing B is given by some B-dominant cocharacter $\check{\lambda}$. Such a P is called a *standard parabolic subgroup* of G, with respect to the choice of (T, B).

Another way to classify standard parabolic subgroups is using the subsets of simple roots Δ. For any dominant cocharacter $\check{\lambda}$, let $\Delta_{\check{\lambda}} \subset \Delta$ be the subset consisting of simple roots perpendicular to $\check{\lambda}$. Then it is well-known that all roots that are linear combinations of ones in $\Delta_{\check{\lambda}}$ form a root subsystem $\Phi_{\check{\lambda}}$ of Φ, and the roots in $P(\check{\lambda})$ is simply the union $\Phi_+ \cup \Phi_{\check{\lambda}}$, and we have a bijection between subsets of Δ and parabolic subgroups containing B.

4.2. Now we consider a general reductive group scheme G over curve X.

Definition 4.3. A parabolic subgroup P of G is a smooth closed subgroup scheme such that over any geometric point $x = \operatorname{Spec} k(x)$ of X the quotient G_x/P_x is projective over $k(x)$.

It is known that G/P is representable by a smooth projective scheme over X. However, in general P may not contain a Borel subgroup scheme of G because the latter may simply not exist. To obtain a good description of parabolic subgroups, we first consider the case where G is an inner twist of the split form $\mathbf{G}$ by a $\mathbf{G}$-torsor E. In this case, we may identify the canonical based root data of $\mathbf{G}$ and G.

Let $\mathbf{B}$ be a split Borel subgroup of $\mathbf{G}$. Suppose that E admits a $\mathbf{B}$-reduction, in other words, we may find a $\mathbf{B}$-torsor $E_{\mathbf{B}}$ that induces E. Then the group

$$B = E_{\mathbf{B}} \times^{\mathbf{B}} \mathbf{B} \subset G = E_{\mathbf{B}} \times^{\mathbf{B}} \mathbf{G}$$

is a Borel subgroup of G. Here $\mathbf{B}$ acts on both $\mathbf{B}$ and $\mathbf{G}$ by conjugation. Conversely, any Borel subgroup $B \subset G$ induces a $\mathbf{B}^{\mathrm{ad}}$-torsor $E_{\mathbf{B}}^{\mathrm{ad}}$ being the sheaf of isomorphisms from $(\mathbf{G}, \mathbf{B})$ to (G, B) inducing identity map on the canonical based root datum. The preimage of $E_{\mathbf{B}}^{\mathrm{ad}}$ under the natural map $E \to \underline{\mathrm{Isom}}(\mathbf{G}, G)$ is then a $\mathbf{B}$-reduction of E. It is then easy to see that it gives a bijective correspondence

$$\{\text{Borel subgroups } B \subset G\} \xleftrightarrow{\sim} \{\mathbf{B}\text{-reductions of } E\}.$$

Similarly, for any parabolic subgroup $P \subset G$, one can always find some standard parabolic $\mathbf{P} \subset \mathbf{G}$ and a $\mathbf{P}$-reduction $E_{\mathbf{P}}$ of E such that $G = E_{\mathbf{P}} \times^{\mathbf{P}} \mathbf{G}$. Such $\mathbf{P}$ is determined by the combinatorial data associated to P (c.f. Definition 4.7), and $E_{\mathbf{P}}$ is obtained by first considering the subsheaf of $\underline{\mathrm{Isom}}_X((\mathbf{G}, \mathbf{P}), (G, P))$ fixing the canonical based root datum, and then take the preimage in E. So we have a bijective correspondence

$$\{\text{Parabolic subgroups } P \subset G\} \xleftrightarrow{\sim} \coprod_{\mathbf{P}} \{\mathbf{P}\text{-reductions of } E\}.$$

where $\mathbf{P}$ ranges over all standard parabolic subgroups of $\mathbf{G}$.

Remark 4.4. One can also try to write down E as a cocycle in $\mathbf{G}$, and then work out the $\mathbf{P}$-cocycle for P. Ultimately it boils down to the fact that the normalizer of P in G is P itself. We leave the details to the reader.

4.5. If we utilize the fact that X is a curve and assume E is generically trivial, the whole process above can be vastly simplified. Indeed, by assumption $\mathbf{G}$ and G are generically isomorphic, and the quotient $E/\mathbf{P}$ admits a generic section. Since $E/\mathbf{P}$ is projective over X, any generic section of $E/\mathbf{P} \to X$ extends to a section over the whole X. Here we use the valuative criteria for properness and the fact that X is a curve.

If furthermore we fix once and for all a generic section of $E/\mathbf{B}$, hence a fixed Borel $B \subset G$, then the parabolic subgroups $P \subset G$ containing such B correspond bijectively to standard parabolic subgroups of $\mathbf{G}$. Therefore, the classification of parabolic subgroups of G in this case is exactly the same as the absolute case.

4.6. Finally, going back to the most general case where G is not a generically split inner form, we let $X' \to X$ be a finite étale cover such that $G' = G \times_X X'$ is a generically split inner form. Any parabolic subgroup P pullback to a parabolic P' of G', hence can be classified by the combinatorial data attached to some Borel $B' \subset G'$ over the generic point, which induces a subset of the canonical simple roots, viewed as a trivial finite étale cover of X'.

Over the intersection $X'' = X' \times_X X'$, the Borel B' may not glue, but the canonical based root data still do, and since P' does descend to X by assumption, its combinatorial data must be stable under monodromy and descends to a finite étale cover of X, which is the union of some connected components of the canonical simple roots of G over X.

Definition 4.7. The connected components of canonical simple roots Δ corresponding to P is called the *reductive type of* P, denoted by $\mathbf{rt}(P)$. The pair $\mathbf{t}(P) = (\Delta, \mathbf{rt}(P))$ is called the *type of* P.

Remark 4.8. Note that we actually do not need X to be a curve or X' to be a finite cover in order to define the type of P, because canonical based root datum may be glued over any étale cover.

Remark 4.9. The notion of $\mathbf{rt}(P)$ is simply called the type of P in SGA3, while in [Beh91, Beh95] the same name is given to the complement $\Delta - \mathbf{rt}(P)$. Both have their advantages: $\mathbf{rt}(P)$ tells you the type of the reductive quotient of P, while its complement tells you how the simple roots of a Borel are twisted by monodromy. Therefore, here we try to incorporate both and give the name *type* to the pair $\mathbf{t}(P)$.

4.10. An alternative formulation using quasi-split forms is as follows: let $\mathbf{G}' \to X$ be the quasi-split form of G associated with the canonical based root datum of G, then there is a $(\mathbf{G}')^{\mathrm{ad}}$-torsor $(E')^{\mathrm{ad}}$ such that $G^{\mathrm{ad}} = (E')^{\mathrm{ad}} \times^{(\mathbf{G}')^{\mathrm{ad}}} (\mathbf{G}')^{\mathrm{ad}}$. So we have bijection

$$\{\text{Parabolic subgroups } P \subset G\} \overset{\sim}{\longleftrightarrow} \coprod_{(\mathbf{P}')^{\mathrm{ad}}} \left\{(\mathbf{P}')^{\mathrm{ad}}\text{-reductions of } (E')^{\mathrm{ad}}\right\},$$

where $(\mathbf{P}')^{\mathrm{ad}}$ ranges over standard parabolic subgroups of $(\mathbf{G}')^{\mathrm{ad}}$ (defined as the ones containing the Borel subgroup in the fixed pinning). This formulation is not as elegant as the case of strongly inner form, but it also shows that the center of the group has limited role in characterizing parabolic subgroups unlike the roots. For this reason, we will not use this formulation much in this exposition in favor of the type map below.

4.11. Let $\underline{\mathrm{Par}}(G)$ be the functor sending X-scheme Y to the set of parabolic subgroups of $G \times_X Y$, and let $\mathcal{P}(\Delta)$ to be the functor sending Y to the power set of $\pi_0(\Delta \times_X Y)$. Then it is a fact that both $\underline{\mathrm{Par}}(G)$ and $\mathcal{P}(\Delta)$ are representable by X-schemes. Therefore we can upgrade our notion of (reductive) type into a morphism of functors

$$\mathbf{rt} = \mathbf{rt}_G\colon \underline{\mathrm{Par}}(G) \longrightarrow \mathcal{P}(\Delta).$$

Theorem 4.12. *The map* $\mathbf{rt}$ *is smooth, projective and surjective with integral fibers.*

The result can be found in SGA3, Exposé XXVI. The representability of $\mathcal{P}(\Delta)$ is obvious from faithfully flat descent, but that of $\underline{\mathrm{Par}}(G)$ is trickier because faithfully flat descent fails for projective morphisms in general, and we will not cover the proof in detail.

4.13. So far, we have been characterizing parabolic reductions using some standard parabolic subgroups in a (quasi-)split model. With the help of type map, we can make the characterization intrinsic to group G itself.

Definition 4.14. Let E be a G-bundle over any X-scheme Y. A *parabolic reduction* of E is a closed subscheme $F \subset E$ satisfying the following conditions:

(1) F is smooth over Y, and
(2) there exists an étale cover $Y' \to Y$ and a parabolic subgroup $P \subset G$ over Y' such that F is stable under P and is a P-torsor.

It is clear that if Y is connected, then P must have constant type, viewed as a section of the sheaf $\mathcal{P}(\Delta)$ over Y, even though P may not be defined over Y (for example, it is possible to have Borel reduction of a G-bundle without having a Borel in G at all). The implication is that it is possible to talk about parabolic reductions of a given type without having to worry about the existence of parabolic subgroups. As a result, we have the following theorem:

Theorem 4.15. *Let E be a G-torsor and $\mathbf{p}$ a given parabolic type. Then we have bijection*

$$\{\textit{Parabolic subgroups of } {}^{E}G \textit{ with type } \mathbf{t}(P) = \mathbf{p}\} \xleftrightarrow{\sim} \{\textit{Parabolic reductions of } E \textit{ with type } \mathbf{p}\},$$

where $\mathcal{P}(\Delta_G)$ is identified with $\mathcal{P}(\Delta_{{}^E G})$ through the canonical isomorphism between their canonical based root data.

5. Stability Conditions

In this section we define (semi-)stability conditions for reductive group G and G-bundles, and compare it with slope stability defined by Mumford. Afterwards, we will state the first main result of Behrend about canonical parabolic subgroups.

5.1. Let G be a smooth group scheme of finite type over X. We have the vector bundle $\mathfrak{g}$ being the Lie algebra of G. The *numerical degree* of G is defined as the degree of its Lie algebra as a vector bundle:

$$\deg^{\#} G \coloneqq \deg \mathfrak{g}.$$

Lemma 5.2. *If G is reductive, then $\deg^{\#} G = 0$.*

Proof. Using the results in § 3, we may replace X by a finite étale cover (being degree 0 is preserved by taking finite étale covers) so that G is an inner form of the split group $\mathbf{G}$. Since $\mathfrak{g} = \mathfrak{z}_G \oplus \mathfrak{g}^{\mathrm{sc}}$, where $\mathfrak{z}_G$ is the Lie algebra of Z_G, and conjugation action has no effect on the center, we see that $\mathfrak{z}_G$ is a trivial vector bundle and thus $\deg \mathfrak{z}_G = 0$. The Killing form on $\mathfrak{g}^{\mathrm{sc}}$ identifies $\mathfrak{g}^{\mathrm{sc}}$ with its dual, hence we also have $\deg \mathfrak{g}^{\mathrm{sc}} = 0$. ∎

Corollary 5.3. *For any affine smooth group scheme P of finite type over X, we have*

$$\deg^{\#} P = \deg^{\#} \mathrm{R_u}(P),$$

where $\mathrm{R_u}(P)$ is the unipotent radical of P.

Definition 5.4. The *degree of instability* of reductive group G is the supremum

$$\operatorname{ideg} G \coloneqq \sup_P \{\deg^{\#} P\},$$

where P ranges over all parabolic subgroups of G. If $\operatorname{ideg} G = 0$, then G is called *semi-stable*, and if in addition $\deg^{\#} P < 0$ for all proper parabolic subgroups P, then G is called *stable*. Let $\mathrm{Bun}_G^{\mathrm{sst}}$ (resp. $\mathrm{Bun}_G^{\mathrm{st}}$) be the full subgroupoid of $\mathrm{Bun}_G(\bar{k})$ of semi-stable (resp. stable) bundles.

Remark 5.5. Using Riemann-Roch theorem, it is easy to see that $\operatorname{ideg} G$ is necessarily finite. We leave it to the reader.

Remark 5.6. We shall see later that $\mathrm{Bun}_G^{\mathrm{sst}}$ (resp. $\mathrm{Bun}_G^{\mathrm{st}}$) is the $\bar{k}$-points of an open substack of Bun_G, which we shall denote using the same notation.

5.7. Given a G-bundle E, we have the associated inner twist ${}^E G$ of G

$${}^E G \coloneqq \mathrm{Ad}(E) = E \times^G G,$$

whose Lie algebra is the adjoint bundle ${}^E\mathfrak{g} = \mathrm{ad}(E) = E \times^G \mathfrak{g}$. We define the *numerical degree* of E to be $\deg^{\#} {}^E G$, and similarly we have the notions of (semi-)stability and degree of instability for E.

5.8. Let $E_P \subset E$ be a parabolic reduction of E. Note that in general there may not exist any parabolic subgroup $P \subset G$ such that E_P is a P-bundle, but it does corresponds to a unique parabolic subgroup ${}^E P \subset {}^E G$, see the end of § 4. We call E_P (semi-)stable if ${}^E P / \mathrm{R_u}({}^E P)$ is. In the case where such $P \subset G$ does exist, it is the same as saying $E_P / \mathrm{R_u}(P)$ as $M_P = P / \mathrm{R_u}(P)$-bundle is (semi-)stable. Similarly we can define the degree of instability and so on.

5.9. Now we use GL_2 over $\mathbb{P}^1$ as an example to see the equivalence between the stability conditions here and Mumford's slope conditions. The category of GL_2-torsors is equivalent to the category of rank-2 vector bundles.

Theorem 5.10 (Grothendieck)**.** *Every vector bundle E of rank n on $\mathbb{P}^1$ splits into a direct sum of line bundles*

$$E \cong \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_n),$$

where $d_1 \geq d_2 \geq \cdots \geq d_n$ are integers uniquely determined by E.

Proof. By Serre's vanishing theorem, $E(n)$ is generated by global sections for sufficiently large n. Also by Serre duality, $E(n)$ has no non-zero global section for sufficiently small n. Therefore we can find $n \in \mathbb{Z}$ to be the smallest integer such that $E(n)$ has a non-zero global section, which corresponds to a map

$$s\colon \mathcal{O}_{\mathbb{P}^1} \to E(n).$$

The cokernel of s must be torsion-free, hence locally-free since $\mathbb{P}^1$ is a smooth curve, otherwise it contradicts with minimality of n. Thus we have exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow E(n) \longrightarrow E' \longrightarrow 0.$$

After twisting by $\mathcal{O}_{\mathbb{P}^1}(-1)$, we have that

$$\mathrm{H}^0(\mathbb{P}^1, E'(-1)) \simeq \mathrm{H}^0(\mathbb{P}^1, E(n-1)) = 0,$$

because (and this is what is special about $\mathbb{P}^1$)

$$\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0.$$

This shows that $E'(-l)$ has no non-zero sections for all $l > 0$, because otherwise by tensoring with a non-zero section of $\mathcal{O}_{\mathbb{P}^1}(l-1)$, which does exist, one would obtain a non-zero section of $E'(-1)$, a contradiction. By Serre duality, we have

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^1}}(E', \mathcal{O}_{\mathbb{P}^1}) \simeq \mathrm{H}^0(\mathbb{P}^1, E'(-2)) = 0.$$

Thus $E(n)$ splits into a direct sum of $\mathcal{O}_{\mathbb{P}^1}$ and E', and we are done by induction. ∎

Going back to our GL_2-example. Any proper parabolic subgroup of GL_2 is a Borel subgroup, and a Borel-reduction of a vector bundle E of rank 2 is the same as giving a full flag of vector bundles, in other words, a short exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow E/F \longrightarrow 0,$$

where F and E/F are line bundles. Suppose $E = \mathcal{O}(n) \oplus \mathcal{O}(m)$ (for convenience we will write $\mathcal{O}_{\mathbb{P}^1}$ simply as $\mathcal{O}$ in this example), and $F = \mathcal{O}(a)$, where a, m, n are some integers and $n \geq m$.

The slope of E is simply

$$\mu(E) = \frac{n+m}{2},$$

and $\mu(F) = a$. The requirement that E/F is locally-free is the same as saying the inclusion $s\colon F \to E$ restricts to an injective (i.e., non-zero) map at each closed point $x \in X$. This means that there can be two possibilities:

(1) $a = n$ and F is the first direct factor $\mathcal{O}(n)$, or
(2) $a \leq m$, and s is given by two sections $f \in \mathcal{O}(n-a)$ and $g \in \mathcal{O}(m-a)$ such that f and g have no common zeros.

In the first case $\mu(F) = a = n \geq \mu(E)$ and in the second case $\mu(F) \leq \mu(E)$. So we see that in the sense of slope stability, E will never be stable, and semi-stable if and only if $n = m$.

Remark 5.11. A subtle difference here compared to § 1 must be noted: here we only consider those subsheaves F such that E/F is locally-free, but in § 1 it is not required. However, it is easy to see that either way they give the same definition of (semi-)stability when X is a curve and we leave the details to the reader.

On the other hand, we have the (semi-)stability defined using numerical degrees. The group ${}^E G$ in this case is simply the $\mathcal{O}$-linear automorphism group $\underline{\mathrm{Aut}}_{\mathcal{O}}(E)$ of E, and the corresponding Borel ${}^E B$ is the sheaf of stabilizers

$$ {}^E B = \mathrm{Stab}_{{}^E G}(F) \subset {}^E G, $$

where F is viewed as a subsheaf of E via inclusion s. Similarly, we have the Lie algebra of ${}^E B$ given by

$$ {}^E \mathfrak{b} = \mathrm{Stab}_{\mathrm{ad}(E)}(F). $$

In fact, by Corollary 5.3, we only need to consider its nilpotent radical ${}^E\mathfrak{u}$, which is represented by the subsheaf of matrices

$$ A \in \begin{pmatrix} \mathcal{O} & \mathcal{O}(n-m) \\ \mathcal{O}(m-n) & \mathcal{O} \end{pmatrix} $$

such that $A \cdot E \subset F$ and $A \cdot F = 0$. Since locally over $\mathbb{P}^1$, f and g are represented by polynomials that are coprime, we can find polynomials α and β, viewed as respective local sections of $\mathcal{O}(a-n)$ and $\mathcal{O}(a-m)$, such that

$$ \alpha f + \beta g = 1. $$

then ${}^t(\beta, -\alpha)$ is linearly independent of ${}^t(f,g)$ at every closed point where they are defined. Thus we have $A \cdot {}^t(\beta, -\alpha) = u\,{}^t(f,g)$ for some u which is a local section of $\mathcal{O}(2a-m-n)$. The map $A \mapsto u$ clearly does not depends on the choice of the pair (α, β), because any another pair will only differ by a multiple of $(g, -f)$, and ${}^t(f,g)$ is annihilated by A. This establishes an isomorphism

$$ {}^E\mathfrak{u} \simeq \mathcal{O}(2a-m-n), $$

and thus

$$ \deg^{\#}{}^E B = 2a - m - n. $$

This computation is valid for both cases of F laid out above, therefore if $a = n$ then $\deg^{\#}{}^E B = n - m \geq 0$ and if $a \leq m$ then $\deg^{\#}{}^E B \leq 0$. Therefore the (semi-)stability condition for E using numerical degree is exactly the same as that given by Mumford's slope. In addition, we also see that the degree of instability of E is $n - m$.

We remark that there is something special about the case $a = n$. Suppose $n > m$, then the numerical degree for this specific Borel is at least 1, while for all other Borels ($a \leq m$) the numerical degree is at most -1 and it is easy to see that every negative integer up to $m - n$ can show up this way. Thus the first Borel seems to be isolated from all others. This is not a coincidence, and it is a special case of *Harder-Narasimhan filtration* which we explain in the next section.

6. Harder-Narasimhan Filtration and Stratification

Given group G over curve X, it turns out that there exists a canonical parabolic subgroup P satisfying certain maximazing properties. As a result, given any G-bundle E, there is a canonical parabolic reduction called the *Harder-Narasimhan filtration*. The name *filtration* refers to the fact that if $G = \mathrm{GL}_n$, a parabolic structure of a G-bundle is the same as a partial flag on the induced vector bundle. We will first give the precise statement about this filtration, and then discuss several equivalent formulations, and finally we will give the main idea of the proof.

6.1. We first introduce the notion of *degree* of a G-bundle E as a refinement of its numerical degree. For this definition, we consider a smooth affine group scheme P over X of finite type, such that étale-locally over X it is a constant group. For example, P may be a parabolic subgroup inside reductive G. Let $\mathbb{X}(P)$ be the sheaf of its multiplicative characters. Then $\mathbb{X}(P)$ is étale-locally a finitely generated abelian group, hence an étale group scheme over X. It is an at most countable union of finite étale covers of X.

Given a P-bundle E_P, let λ be a connected component of $\mathbb{X}(P)$ (as an open and closed subscheme). Then λ may be viewed as a finite set of characters of P at some point $x \in X$ on which the monodromy acts transitively. The pullback of E_P to λ induces a $\mathbb{G}_{\mathrm{m}}$-torsor using the tautological P-character on λ. Let d_λ

be the degree of λ over X as a covering, then we have an associated vector bundle $E_P(\lambda)$ of rank d_λ on X. Thus we have map

$$\begin{aligned} \deg E_P \colon \pi_0(\mathbb{X}(P)) &\longrightarrow \mathbb{Z} \\ \lambda &\longmapsto \deg E_P(\lambda). \end{aligned}$$

In addition, we may "sum up the characters" in λ (viewed as a finite set of local characters stable under monodromy), and obtain a global character $\Sigma_\lambda \colon P \to \mathbb{G}_{\mathrm{m}}$, in other words, a global section of $\mathbb{X}(P)$. This corresponds to a connected component of $\mathbb{X}(P)$ isomorphic to X, and we have

$$\deg E_P(\lambda) = \deg E_P(\Sigma_\lambda).$$

Therefore $\deg E_P$ is completely determined by its values on $\Gamma(X, \mathbb{X}(P)) = \mathrm{Hom}_X(P, \mathbb{G}_{\mathrm{m}})$ and is clearly additive, thus may be viewed as an element of the abelian dual

$$\deg E_P \in \mathrm{Hom}_X(P, \mathbb{G}_{\mathrm{m}})^\vee.$$

Definition 6.2. The element $\deg E_P \in \mathrm{Hom}_X(P, \mathbb{G}_{\mathrm{m}})^\vee$ is called the *degree* of E_P.

Lemma 6.3. *Let $X' \to X$ be a finite étale cover of degree m, and let E'_P be the pullback of E_P to X'. Then the image of* $\deg E'_P$ *in* $\mathrm{Hom}_X(P, \mathbb{G}_{\mathrm{m}})^\vee$ *is equal to $m \deg E_P$.*

Proof. Let $\lambda \in \mathrm{Hom}_X(P, \mathbb{G}_{\mathrm{m}})$, then $\deg E'_P(\lambda) = m \deg E_P(\lambda)$. ∎

6.4. We may recover numerical degree from degree in a relative sense: the adjoint action of P induces character

$$2\rho_P \colon P \longrightarrow \mathrm{GL}(\mathfrak{p}) \xrightarrow{\det} \mathrm{GL}(\det \mathfrak{p}) = \mathbb{G}_{\mathrm{m}}.$$

Similarly, given E_P, we have the associated group ${}^E P$, and thus a character $2\rho_{({}^E P)}$, and we may also view P as a twist of ${}^E P$ by an ${}^E P$-bundle E_P^-.

Proposition 6.5. *We have equalities*

$$\deg^\#({}^E P) - \deg^\#(P) = \deg E_P(2\rho_P) = -\deg E_P^-(2\rho_{({}^E P)}).$$

In particular, if P is a standard parabolic of a split reductive group G, then $\deg^\#({}^E P) = \deg E_P(2\rho_P)$.

Proof. By definition we have

$$\mathrm{ad}(E_P) = E_P \times^P \mathfrak{p},$$

so its determinant bundle is just the line bundle induced by the action of P on $\det \mathfrak{p}$ through character $2\rho_P$. This is the same as line bundle $E_P(2\rho_P) \otimes_{\mathcal{O}_X} (\det \mathfrak{p})$, and the degree of $\det \mathfrak{p}$ is $\deg^\# P$ by definition. ∎

Remark 6.6. In our discussion above, if we forgo the assumption that P is locally constant, then $\mathbb{X}(P)$ will become a countable union of finite but not necessarily étale cover of X, and it may have singular components. In this case the degree map can still be defined, but the story may be more complicated due to $\mathrm{Hom}_X(P, \mathbb{G}_{\mathrm{m}})$ not capturing the information of ramifications. For example, P can be the regular centralizer group scheme arising from Hitchin fibrations. In any case we do not pursue it here.

6.7. It is not hard to see that the definition of $\deg E_P$ may be generalized to P-bundle over $X \times \operatorname{Spec} R$ for any k-algebra R, and since $\mathrm{Hom}_X(P, \mathbb{G}_{\mathrm{m}})^\vee$ is discrete, there is a decomposition of Bun_P into open and closed substacks

$$\mathrm{Bun}_P = \coprod_{\nu \in \mathrm{Hom}_X(P, \mathbb{G}_{\mathrm{m}})^\vee} \mathrm{Bun}_P^\nu,$$

where $\mathrm{Bun}_P^\nu(R)$ consists of P-bundles on $X \times \operatorname{Spec} R$ such that for each geometric point x of $\operatorname{Spec} R$, its restriction to $X \times \{x\}$ has degree ν.

Remark 6.8. Be cautious that each Bun_P^ν may be disconnected. For example, if $P = \mathrm{PGL}_2$, then Bun_P has two connected components, but clearly $\mathrm{Hom}_X(P, \mathbb{G}_{\mathrm{m}})^\vee$ is trivial.

Let $U_P = \mathrm{R_u}(P)$ be the unipotent radical, and $M_P = P/U_P$ be the reductive quotient, then we have

$$\mathrm{Bun}_P \longrightarrow \mathrm{Bun}_{M_P}.$$

Let $\mathrm{Bun}_P^{\mathrm{sst}}$ be the preimage of $\mathrm{Bun}_{M_P}^{\mathrm{sst}}$ (at this point only as a $\bar{k}$-groupoid, see Remark 5.6), and we have thus

$$\mathrm{Bun}_P^{\mathrm{sst}} = \coprod_{\nu} \mathrm{Bun}_P^{\nu,\mathrm{sst}}.$$

Similarly, we also have the stable version.

6.9. Now we restrict to the case when P is a parabolic subgroup of a reductive group G. The canonical maximal torus of G may be identified with the canonical maximal torus of M_P because they can be canonically identified étale-locally, and similarly the canonical based root datum of M_P may be viewed as a sub-datum in that of G. Let $\mathbb{X}$ be the character lattice in such root datum. Then we have natural maps of abelian X-sheaves

$$\mathbb{X}(P) \xleftarrow{\sim} \mathbb{X}(M_P) \longrightarrow \mathbb{X} \longrightarrow \mathbb{X}(Z_{M_P}),$$

such that after tensoring with $\mathbb{Q}$, we have isomorphisms

$$\mathbb{X}(P)_{\mathbb{Q}} \xleftarrow{\sim} \mathbb{X}(M_P)_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{X}(Z_{M_P})_{\mathbb{Q}}. \tag{6.9.1}$$

Using the discrete topology on $\mathbb{Q}$, then the above sheaves are represented by a countable union of finite étale covers of X.

Let $\mathbf{t}(P) = (\Delta, \mathbf{rt}(P))$ be the type of P. Each connected component α of $\Delta - \mathbf{rt}(P) \subset \mathbb{X}$ may be viewed as a finite set of local characters of the canonical maximal torus twisted by monodromy. Taking the image of α in $\mathbb{X}(Z_{M_P})$ and then its preimage in $\mathbb{X}(P)_{\mathbb{Q}}$, it may be viewed as a set of local rational characters of P twisted by monodromy. Extending coefficients from $\mathbb{Z}$ to $\mathbb{Q}$, then we may view $\deg E_P$ as an element of

$$\Gamma(X, \mathbb{X}(P))_{\mathbb{Q}}^{\vee},$$

so its evaluation on α makes sense as $\deg E_P(\alpha) \in \mathbb{Q}$.

6.10. Here we add a quick comment on the *slope map* defined in [Sch15]. It is really just a re-packaging of some previous notions, but does give a more explicit connection to Mumford's slope for vector bundles. Indeed, (6.9.1) induces a map

$$\Gamma(X, \mathbb{X})_{\mathbb{Q}} \longrightarrow \mathrm{Hom}_X(P, \mathbb{G}_{\mathrm{m}})_{\mathbb{Q}},$$

hence a map

$$\phi_P \colon \mathrm{Hom}_X(P, \mathbb{G}_{\mathrm{m}})_{\mathbb{Q}}^{\vee} \longrightarrow \Gamma(X, \mathbb{X})_{\mathbb{Q}}^{\vee},$$

which is called the *slope map*. When $G = \mathrm{GL}_n$ and E is a vector bundle of rank n with degree d_E, let $P = G$, then $\mathbb{X}(G) \cong \mathbb{Z}$ generated by the usual determinant, and $\mathbb{X} \cong \mathbb{Z}^n$. One can easily see that

$$\phi_G(d_E) = (\mu(E), \ldots, \mu(E)) \in \mathbb{Q}^n.$$

In addition, if

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_m = E$$

is a partial flag of vector bundles in (i.e., a parabolic reduction of) E with degree ν_P, then one also show that

$$\phi_P(\nu_P) = (\mu(E_1), \ldots, \mu(E_1), \ldots, \mu(E_m/E_{m-1}), \ldots, \mu(E_m/E_{m-1})) \in \mathbb{Q}^n,$$

with each $\mu(E_i/E_{i-1})$ repeating $\mathrm{rk}(E_i/E_{i-1})$ times. See [Sch15] for details.

Definition 6.11. An element $\nu \in \mathrm{Hom}_X(P, \mathbb{G}_{\mathrm{m}})^{\vee}$ is called *dominant* if $\langle \alpha, \nu \rangle \geq 0$ for every connected component $\alpha \in \Delta - \mathbf{rt}(P)$. It is called *dominant regular* if the inequalities are strict. Denote the subset of dominant (resp. dominant regular) elements of $\mathrm{Hom}_X(P, \mathbb{G}_{\mathrm{m}})^{\vee}$ by $\mathrm{Hom}_X(P, \mathbb{G}_{\mathrm{m}})^{\vee}_{\geq 0}$ (resp. $\mathrm{Hom}_X(P, \mathbb{G}_{\mathrm{m}})^{\vee}_{>0}$).

If $P' = gPg^{-1}$ for some $g \in G$, then we have identification between $\mathbb{X}(P)$ and $\mathbb{X}(P')$ induced by g. Because the normalizer of P in G is P itself, we see that this identification is independent of the choice of g, and so we have a *canonical* identification

$$\mathbb{X}(P) \simeq \mathbb{X}(P').$$

Taking inductive limits over all conjugates of P and denote it by $\mathbb{X}(\mathbf{t}(P))$, where $\mathbf{t}(P)$ is the type of P. Conversely, for each parabolic type $\mathbf{p}$, there is an associated abelian sheaf on X, denoted by $\mathbb{X}(\mathbf{p})$.

The isomorphism between P and P' also induces isomorphism between Bun_P and $\mathrm{Bun}_{P'}$, and our discussion just now implies that the degree of a P-bundle does not depend on the group P at all but only on its type. Similarly, the notion of (regular) dominance makes sense for any element in the $\mathbb{Q}$-dual of $\Gamma(X, \mathbb{X}(\mathbf{p}))$. As a consequence, it makes sense to define the *degree of a parabolic reduction* as an element of the $\mathbb{Z}$-dual of $\Gamma(X, \mathbb{X}(\mathbf{p}))$ and its (regular) dominance.

6.12. In the context of $G = \mathrm{GL}_n$ (i.e., vector bundles), there is the notion of canonical flag, which we calculated in § 4 for GL_2. More precisely, for any vector bundle E on X, there is a unique filtration of vector bundles

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_m \subset E,$$

such that the associated graded bundle is semi-stable and $\mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(E/E_m)$. The analogue of flags for arbitrary reductive group is parabolic reduction. Therefore we have the following theorem by Behrend:

Theorem 6.13 (Harder-Narasimhan Filtration)**.** *Let E be a G-bundle, there is a unique parabolic subgroup (called the* canonical parabolic subgroup*) ${}^E P \subset {}^E G$ characterized by one of the following equivalent conditions:*

(1) $\deg^{\#} {}^E P = \mathrm{ideg}\, {}^E G$ *and ${}^E P$ is the unique maximal element among all parabolic subgroups of ${}^E G$ with this property.*
(2) *There exists a unique parabolic reduction E_P of E such that E_P is semi-stable with the same type of ${}^E P$ and* $\deg E_P$ *is regular dominant.*

Remark 6.14.
(1) The first characterization in the above theorem is purely stated using group ${}^E G$, while the second is in terms of bundles. The regular dominance condition corresponds to the positivity condition for certain invariants in [Beh95].
(2) Over algebraically closed base field, since X is a curve, one can always realize E_P as a P-bundle for some parabolic subgroup P of G.

6.15. Since the Harder-Narasimhan filtration is indexed by discrete data, one naturally ask whether it induces a stratification of Bun_G. It turns out to be indeed the case, and such stratification of Bun_G is called *Harder-Narasimhan stratification.*

Theorem 6.16 (Harder-Narasimhan Stratification)**.** *There is a unique stratification of* Bun_G *into locally closed substacks over $\bar{k}$*

$$\mathrm{Bun}_G = \coprod_{\substack{(\mathbf{p},\nu) \\ \nu \in \Gamma(X_{\bar{k}}, \mathbb{X}(\mathbf{p}))^{\vee}_{>0}}} \mathrm{Bun}_G^{\mathbf{p},\nu},$$

such that for any $E \in \mathrm{Bun}_G^{\mathbf{p},\nu}(\bar{k})$, the Harder-Narasimhan filtration of E has parabolic type $\mathbf{p}$ and degree ν. Moreover, each stratum is an at most countable disjoint union of algebraic stacks of finite type over $\bar{k}$, and the strata corresponding to semi-stable G-bundles are open in Bun_G*, .*

We will not cover the proof of this theorem in detail, but the important step is to show certain semi-continuity result regarding the combinatorial data of the Harder-Narasimhan filtration over a discrete valuation ring. With such result one can then prove that each $\mathrm{Bun}_G^{\mathbf{p},\nu}$, first defined at $\bar{k}$-level, is locally closed topologically. The finite type statement follows from some standard reduction to the case of split group and the properties of Bun_B where B is a Borel subgroup of a split group.

6.17. Although we already obtained Harder-Narasimhan filtration in general, it would be nice to connect each stratum with some more familiar objects: for example, the open strata consist of semi-stable G-bundles. Such "refinement", not in the sense of stratification, is most conveniently stated when G is split. If G is not split, one has to state the result for a slightly coarser stratification, which we will see is not a serious downgrade because it still preserves the information about parabolic types.

6.18. To start, recall that for reductive group scheme G we have an associated quasi-split form $\mathbf{G}'$ and its adjoint quotient $(\mathbf{G}')^{\mathrm{ad}}$. There exists some $(\mathbf{G}')^{\mathrm{ad}}$-torsor $(\mathbf{E}'_\sigma)^{\mathrm{ad}}$ such that

$$G^{\mathrm{ad}} \cong (\mathbf{E}'_\sigma)^{\mathrm{ad}} \times^{(\mathbf{G}')^{\mathrm{ad}}} (\mathbf{G}')^{\mathrm{ad}} = \mathrm{Aut}_{(\mathbf{G}')^{\mathrm{ad}}}((\mathbf{E}'_\sigma)^{\mathrm{ad}}).$$

Therefore we have isomorphism of functors

$$\begin{aligned} \sigma\colon\ \mathrm{Bun}_{G^{\mathrm{ad}}} &\longrightarrow \mathrm{Bun}_{(\mathbf{G}')^{\mathrm{ad}}} \\ E^{\mathrm{ad}} &\longmapsto (E'_0)^{\mathrm{ad}} \times^{G^{\mathrm{ad}}} E^{\mathrm{ad}}, \end{aligned}$$

where the inverse is $(\mathbf{E}')^{\mathrm{ad}} \mapsto \underline{\mathrm{Isom}}_{(\mathbf{G}')^{\mathrm{ad}}}((\mathbf{E}')^{\mathrm{ad}}, (\mathbf{E}'_\sigma)^{\mathrm{ad}})$. Note that σ depends on the choice of the torsor $(\mathbf{E}'_\sigma)^{\mathrm{ad}}$ which is not canonical (for example, if $\mathbf{G}' = \mathbf{G}$ is split, then $\mathrm{Out}(\mathbf{G}^{\mathrm{ad}})$ acts on the set of possible choices simply transitively; in general it is replaced by the global sections of $\mathrm{Out}((\mathbf{G}')^{\mathrm{ad}})$, the latter being an inner form of group $\mathrm{Out}(\mathbf{G}^{\mathrm{ad}})$).

On the other hand, we also have smooth map

$$\mathrm{Bun}_G \longrightarrow \mathrm{Bun}_{G^{\mathrm{ad}}}.$$

Through σ, we obtain a smooth map

$$\sigma_G\colon\ \mathrm{Bun}_G \longrightarrow \mathrm{Bun}_{(\mathbf{G}')^{\mathrm{ad}}}.$$

Theorem 6.19. *[Harder-Narasimhan stratification, alternative version] Let G be a reductive group scheme over X and let $(\mathbf{G}')^{\mathrm{ad}}$ be the quasi-split form of adjoint group G^{ad}. Then we have:*

1. *The semi-stable locus $\mathrm{Bun}_G^{\mathrm{sst}}$ is open in Bun_G. In particular, it is true for the reductive quotient $M_P = P/U_P$ of a parabolic subgroup P of G, hence $\mathrm{Bun}_P^{\mathrm{sst}} \subset \mathrm{Bun}_P$ is also an open embedding of algebraic stacks.*
2. *The semi-stable locus is an at most countable disjoint union of algebraic stacks of finite type over k.*
3. *For each standard parabolic subgroup $(\mathbf{P}')^{\mathrm{ad}} \subset (\mathbf{G}')^{\mathrm{ad}}$ and each $\nu \in \mathrm{Hom}_X((\mathbf{P}')^{\mathrm{ad}}, \mathbb{G}_m)^\vee_{>0}$, there exists a maximal locally closed substack $\mathrm{Bun}_G^{(\mathbf{P}')^{\mathrm{ad}},\nu} \subset \mathrm{Bun}_G$ such that its image $\mathrm{Bun}_{(\mathbf{G}')^{\mathrm{ad}}}^{(\mathbf{P}')^{\mathrm{ad}},\nu}$ under σ_G is the same as the image of $\mathrm{Bun}_{(\mathbf{P}')^{\mathrm{ad}}}^{\nu,\mathrm{sst}}$, and such that the map*
$$\mathrm{Bun}_{(\mathbf{P}')^{\mathrm{ad}}}^{\nu,\mathrm{sst}} \longrightarrow \mathrm{Bun}_{(\mathbf{G}')^{\mathrm{ad}}}^{(\mathbf{P}')^{\mathrm{ad}},\nu}$$
is surjective finite radicial, and in particular a universal homeomorphism. If the characteristic of base field k is not among a finite set of "bad primes" (depending on G), then the said map is an isomorphism of stacks.
4. *The collection of all possible $((\mathbf{P}')^{\mathrm{ad}}, \nu)$ determines a stratification of Bun_G by $\mathrm{Bun}_G^{(\mathbf{P}')^{\mathrm{ad}},\nu}$. In other words, these strata exhaust Bun_G.*

Remark 6.20. The stratification in Theorem 6.19 does not depend on the choice of σ_G, but the indexation by $((\mathbf{P}')^{\mathrm{ad}}, \nu)$ does. It is also clear that each stratum is a disjoint union of some strata in Theorem 6.16 as open and closed substacks.

6.21. For split groups, or more generally for those G that is induced by a $\mathbf{G}'$-torsor $\mathbf{E}'_\sigma$ (not just a $(\mathbf{G}')^{\mathrm{ad}}$-torsor), Theorem 6.19 can be improved to better match Theorem 6.16.

Theorem 6.22. *Let G be induced by a $\mathbf{G}'$-torsor. Then we have the following refinement of Theorem 6.19: for each standard parabolic subgroup $\mathbf{P}' \subset \mathbf{G}'$ and each $\nu \in \mathrm{Hom}_X(\mathbf{P}', \mathbb{G}_m)^\vee_{>0}$, there exists a maximal locally closed substack $\mathrm{Bun}_G^{\mathbf{P}',\nu} \subset \mathrm{Bun}_G$ such that its image $\mathrm{Bun}_{\mathbf{G}'}^{\mathbf{P}',\nu}$ under σ_G is the same as the image of $\mathrm{Bun}_{\mathbf{P}'}^{\nu,\mathrm{sst}}$, and such that the map*

$$\mathrm{Bun}_{\mathbf{P}'}^{\nu,\mathrm{sst}} \longrightarrow \mathrm{Bun}_{\mathbf{G}'}^{\mathbf{P}',\nu}$$

is surjective finite radicial, and in particular a universal homeomorphism. If the characteristic of base field k is not among a finite set of "bad primes" (depending on G), then the said map is an isomorphism of stacks.

Remark 6.23. It is also easy to see that the strata in Theorem 6.22 are in bijection with those in Theorem 6.16.

7. Complementary Polyhedra

In this section we do not cover any new results. Rather, we would like to include Behrend's notion of complementary polyhedra and comment on its relevance in the whole story. One consequence of Behrend's polyhedra is that it allows us to define so-called "weighted stability", which is important in the study of automorphic trace formulae. We will refrain from covering weighted stability itself, however.

7.1. For this section we will assume that the base field k is algebraically closed, and the group G is generically split. Let K be the function field of X, then since there exists a split maximal torus in G_K, every maximal torus in G_K is split.

Fix a maximal torus $T_K \subset G_K$, there is a natural bijection between the Weyl chambers of the root system of T_K in G_K and the Borel subgroups of G_K containing T_K. Since the sheaf of flag varieties of G is representable by a projective X-scheme and since X is a curve, any Borel subgroup $B_K \subset T_K$ extends to a Borel subgroup $B \subset G$ by valuative criteria for properness.

7.2. The canonical based root datum of G is necessarily constant and is identified with the based root datum of G_K determined by (T_K, B_K). This means that G is an inner twist of $\mathbf{G}$ induced by a $\mathbf{G}^{\mathrm{ad}}$-bundle E^{ad} such that E^{ad}_K is trivial, and B corresponds to a $\mathbf{B}$-reduction $E^{\mathrm{ad}}_{\mathbf{B}}$. Let ν_B be the degree of $E^{\mathrm{ad}}_{\mathbf{B}}$, which can be seen as an element in $\check{\mathbb{X}}^{\mathrm{ad}} = \check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}})$.

Using the identification between $\check{\mathbb{X}}^{\mathrm{ad}}$ and $\check{\mathbb{X}}(T_K)$ induced by B_K (which is different for different B), we may view $\nu_B \in \check{\mathbb{X}}(T^{\mathrm{ad}}_K)$. The collection of ν_B can be seen as a map

$$\begin{aligned} d_{T_K}\colon\ \mathrm{cham}(T_K) &\longrightarrow \check{\mathbb{X}}(T^{\mathrm{ad}}_K) \\ B &\longmapsto d_{T_K}(B) = \nu_B, \end{aligned}$$

where $\mathrm{cham}(T_K)$ denotes the set of Borel subgroups in G_K containing T_K, or equivalently the set of Weyl chambers in the root system of T_K in G_K.

Definition 7.3. Let $R = (V, \Phi, \check{V}, \check{\Phi})$ be an abstract root system where V is the $\mathbb{Q}$-vector space spanned by the set of roots Φ and $\check{V}$ by coroots $\check{\Phi}$. Let $\mathrm{cham}(R)$ be the set of Weyl chambers in R. A *complementary polyhedron* for R is a map

$$d\colon\ \mathrm{cham}(R) \longrightarrow \check{V}$$

such that

(1) If $\lambda \in \bar{C} \cap \overline{C'}$ for two Weyl chambers C and C', then
$$\langle \lambda, d(C)\rangle = \langle \lambda, d(C')\rangle.$$
(2) If C and C' differ by a reflection determined by a root α, and suppose α is positive with respect to C and negative with respect to C', then
$$\langle \alpha, d(C)\rangle \le \langle \alpha, d(C')\rangle.$$

Proposition 7.4 ([Beh95, Proposition 6.6])**.** *The map d_{T_K} is a complementary polyhedron for the root system of T_K in G_K.*

7.5. In order to characterize the canonical parabolic subgroup (or equivalently the Harder-Narasimhan filtration) using complementary polyhedra, we need to study these polyhedra purely as combinatorial data.

7.6. We start with just an abstract root system R without any complementary polyhedron. Each pair of roots $\{\pm\alpha\}$ determines a hyperplane $H_\alpha = \check{\alpha}^\perp \subset V$. These hyperplanes induces a stratification of the vector space V into facets so that $x, y \in V$ belong to the same facet P if and only if for any root α, one either has $\langle x, \check{\alpha}\rangle = \langle y, \check{\alpha}\rangle = 0$ or $\langle x, \check{\alpha}\rangle\langle y, \check{\alpha}\rangle > 0$.

There is a partial order on the set of facets such that $P \le Q$ if $Q \subset \bar{P}$. With this order, there is a unique maximal facet $\{0\}$, and the minimal ones are the Weyl chambers. If R is the root system of a split maximal torus T in a reductive group G, then we know there is a bijection between facets and parabolic subgroups of G containing T. Under this bijection, the maximal facet $\{0\}$ corresponds to G itself, and the minimal ones correspond to Borel subgroups containing T. If we partially order parabolic subgroups by inclusion, then the correspondence between facets and parabolic subgroups containing T is order preserving.

7.7. For each facet P, we denote by $\mathrm{cham}(P)$ the set of Weyl chambers $B \le P$. This corresponds to all the Borels containing T that are contained in P in the group case. In particular, $\mathrm{cham}(\{0\}) = \mathrm{cham}(R)$.

For each $B \in \mathrm{cham}(R)$, it determines a set of fundamental weights Π_B. We define

$$\Pi_P \coloneqq \bigcap_{B\in\mathrm{cham}(P)} \Pi_B,$$
$$\Pi'_P \coloneqq \bigcup_{B\in\mathrm{cham}(P)} \Pi_B,$$

and for convenience let $\Pi = \Pi'_{\{0\}} = \bigcup_{B\in\mathrm{cham}(R)} \Pi_B$. Note that by definition $\Pi_B = \Pi'_B$ for $B \in \mathrm{cham}(R)$ and $\Pi_{\{0\}} = \varnothing$. It is easy to see that for each facet P, the set Π_P generates the closure $\overline{P}$ in V as a free monoid over $\mathbb{Q}_{\ge 0}$.

7.8. Let P be a facet of R. Switching the role of V (resp. Φ) and $\check{V}$ (resp. $\check{\Phi}$), we have the dual facet $\check{P}$ of P. Then we have canonical decompositions of vector spaces

$$V = \operatorname{span} P \oplus \check{P}^{\perp},$$
$$\check{V} = \operatorname{span} \check{P} \oplus P^{\perp}.$$

Denote $V_P = \check{P}^{\perp}$ and $\check{V}_P = P^{\perp}$, we define projection maps

$$p\colon V \longrightarrow V_P$$
$$\check{p}\colon \check{V} \longrightarrow \check{V}_P.$$

The pairing $V \times \check{V} \to \mathbb{Q}$ induces a pairing of subspaces

$$(0 \oplus V_P) \times (0 \oplus \check{V}_P) \longrightarrow \mathbb{Q},$$

which is easily seen to be non-degenerate. Using the projection maps p and $\check{p}$, we have a pairing

$$V_P \times \check{V}_P \longrightarrow \mathbb{Q},$$

which identifies $\check{V}_P$ with the dual space of V_P. Note, however, that the diagram

$$\begin{tikzcd} V \times \check{V} \arrow[r] \arrow[d, "p\times\check{p}"'] & \mathbb{Q} \\ V_P \times \check{V}_P \arrow[ur] & \end{tikzcd}$$

does *not* commute, but both diagrams

$$\begin{tikzcd} (0 \oplus V_P) \times \check{V} \arrow[r] \arrow[d, "p\times\check{p}"'] & \mathbb{Q} \\ V_P \times \check{V}_P \arrow[ur] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} V \times (0 \oplus \check{V}) \arrow[r] \arrow[d, "p\times\check{p}"'] & \mathbb{Q} \\ V_P \times \check{V}_P \arrow[ur] & \end{tikzcd}$$

do commute.

7.9. Let $\Phi_P = p(\Phi \cap (0 \oplus V_P))$ and $\check{\Phi}_P = \check{p}(\check{\Phi} \cap (0 \oplus \check{V}_P))$, then

$$R_P = (V_P, \Phi_P, \check{V}_P, \check{\Phi}_P)$$

is also a root system. In the group case, it is the root system of the reductive quotient M_P of P with respect to the same maximal torus T. The map p induces a bijection

$$\{\text{Facets } Q \le P\} \xrightarrow{\sim} \{\text{Facets of } R_P\}$$
$$Q \longmapsto p(Q)$$

This corresponds to the bijection between the set of parabolic subgroups of G containing T that are contained in P and the set of parabolic subgroups of M_P containing T.

7.10. Let d be a complementary polyhedron for R. We will now define so-called *numerical invariant* of a facet P with respect to d, which is secretly corresponding to the notion of degree of parabolic reductions in previous sections.

Let $\lambda \in \Pi_P$ be a generator of the cone $\bar{P}$ and let $\check{\lambda}$ be its dual element (so that $\langle \lambda, \check{\alpha} \rangle = \langle \alpha, \check{\lambda} \rangle$ for all roots α). We define the *elementary set of roots associated with P and λ* to be the set

$$\Phi(P, \lambda) = \left\{ \alpha \in \Phi \;\middle|\; \begin{array}{l} \langle \alpha, \check{\lambda} \rangle = 1, \\ \langle \alpha, \check{\mu} \rangle = 0, \forall \mu \in \Pi_P - \{\lambda\} \end{array} \right\}.$$

If $P = B$ is a Weyl chamber and λ is a fundamental weight, then $\Phi(P, \lambda)$ is a singleton of the unique simple root determined by B and λ.

Lemma 7.11 ([Beh95, Lemma 3.6])**.** *For each $\lambda \in \Pi_P$, we have*

$$\sum_{\alpha \in \Phi(P,\lambda)} \alpha \in \operatorname{span}(P).$$

Using this lemma and the definition of d as a complementary polyhedron, we see that the number

$$n_d(P, \lambda, B) = \sum_{\alpha \in \Phi(P,\lambda)} \langle \alpha, d(B) \rangle$$

does not depend on B if $B \leq P$. Let $n_d(P, \lambda) = n_d(P, \lambda, B)$ for any choice of $B \leq P$. This is the *numerical invariant of P with respect to λ and d*. Using these numerical invariants, we extend d to all facets (previously d is just a map on the minimal facets):

$$d(P) \coloneqq \sum_{\lambda \in \Pi_P} \frac{n_d(P, \lambda)}{\# \Phi(P, \lambda)} \check{\lambda} \in \operatorname{span} \check{P} \subset \check{V}.$$

It is easy to see that for any Weyl chamber B we have

$$d(B) = \sum_{\lambda \in \Pi_B} n_d(B, \lambda) \check{\lambda},$$

so it is indeed an extension of the original d.

7.12. At this point, we make a more explicit comment on the connection between the numerical invariants and the degrees of parabolic reductions. If, as in the beginning of this section, let E^{ad} be the generically trivial $\mathbf{G}^{\mathrm{ad}}$-torsor inducing the inner form G and E_B^{ad} be a $\mathbf{B}^{\mathrm{ad}}$-reduction corresponding to $B \subset G$ such that B_K is a Borel subgroup containing $T_K \subset G_K$, and we still use B to denote the Weyl chamber of B_K in the root system $R = R(G_K, T_K)$ of T_K in G_K, then one has by definition that $d(B) = d_{T_K}(B)$ is the degree of E_B^{ad}.

A facet $P \geq B$ determines a parabolic $P_K \subset G_K$ containing B_K, which, again by projectivity, extends to a parabolic subgroup $P \subset G$. This corresponds to the $\mathbf{P}$-reduction E_P^{ad} of E^{ad} which is the same as $E_B^{\mathrm{ad}} \times^{\mathbf{B}^{\mathrm{ad}}} \mathbf{P}^{\mathrm{ad}}$ where $\mathbf{P} \subset \mathbf{G}$ is a standard parabolic subgroup of the same type as P. One can also show that the degree of E_P^{ad} is equal to $d(P)$. Here we use the natural identification

$$\operatorname{span}(\check{P}) \simeq \operatorname{Hom}_X(\mathbf{P}^{\mathrm{ad}}, \mathbb{G}_{\mathrm{m}})_{\mathbb{Q}}.$$

Moreover, since $\# \Phi(P, \lambda)$ is always positive, the regular dominance condition for degrees is the same as saying the numerical invariants are all positive. All the details in this subsection is left to the readers and they can all be found (implicitly or explicitly) in [Beh95] for example.

7.13. Using the projection maps p and $\check{p}$, the complementary polyhedron d induces a complementary polyhedron d_P for R_P as follows: each Weyl chamber $B_P \in \operatorname{cham}(R_P)$ uniquely corresponds to a Weyl chamber B such that $B \leq P$, and we define

$$d_P(B_P) = \check{p}(d(B)).$$

It is easily checked to satisfy the definition of complementary polyhedra.

For a facet P, we define the *dual polyhedron* $F(P)$ of P as the convex hull of those $d(B)$ for $B \in \operatorname{cham}(P)$, and let $F = F(\{0\})$. One can show that for $P \leq Q$, we have $F_P(p(Q)) = \check{p}(F(Q))$, where F_P means the analogue of F for (R_P, d_P). In particular, $F_P = F_P(\{0\}) = \check{p}(F(P))$.

Definition 7.14. A root system with a complementary polyhedron d is called *semi-stable* if $0 \in F$, and *stable* if 0 is contained in the interior of F (with the usual Archimedean topology on V).

Proposition 7.15. *Let G be an inner twist of $\mathbf{G}$ induced by a generically trivial $\mathbf{G}^{\mathrm{ad}}$-torsor E^{ad}. Then G (equivalently, E^{ad}) is semi-stable if and only if the complementary polyhedron d_{T_K} is semi-stable for all maximal tori $T_K \subset G_K$.*

This result is a consequence of the characterization of canonical parabolic subgroup using complementary polyhedra, which we will state using so-called *special facet*.

Definition 7.16. Let (R, d) be a root system with a complementary polyhedraon. A facet P of R is called *special* with respect to d if the followings hold:

(1) For all $\lambda \in \Pi_P$ we have $n_d(P, \lambda) > 0$.
(2) The induced root system with complementary polyhedron (R_P, d_P) is semi-stable.

Theorem 7.17. *For any (R, d), there exists a unique special facet P, which is also characterized by any of the following conditions:*

(1) $\check{P} \cap F(P) \neq \varnothing$.
(2) $d(P) \in \check{P} \cap F(P)$.
(3) *Let $x \in F$ be the unique closest point of F to $0 \in \check{V}$ (under L^2-norm), then $x \in P$ and $x = d(P)$.*

Proof. See [Beh95]. ∎

Remark 7.18. There is yet another characterization of the special facet using *degrees* of facets. In short, one can define the degree of a facet with respect to a complementary polyhedron, which is essentially the combinatorial counterpart to the numerical degree of a parabolic subgroup. Then the special facet is the maximal element among all facets with maximal degree. See [Beh95] for details.

Theorem 7.19. *Let G be an inner twist of $\mathbf{G}$ induced by a generically trivial $\mathbf{G}^{\mathrm{ad}}$-torsor E^{ad}. Then a parabolic subgroup P of G is canonical if and only if for all maximal torus $T_K \subset G_K$ contained in P_K, the facet of P in $R(G_K, T_K)$ is special with respect to d_{T_K}.*

7.20. Finally, for general group G, one make a finite étale base change $X' \to X$ so that over X' the group G becomes generically split, and one can talk about complementary polyhedra over the generic point of X'. One can certainly try to make sense of complementary polyhedra and special facets as some combinatorial data with Galois action (using the universal finite étale cover of X), but we will not complicate this exposition any further.

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