GEOMETRIC SATAKE: COURSE NOTES FOR V. GINZBURG'S CLASS ON SPRING 2018

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1. March 27

Let Σ be a smooth complete curve over $\mathbb{F}_q$, and Σ^{aff} an affine open subvariety, and let the infinity be the set $\infty = \Sigma - \Sigma^{\mathrm{aff}}$. Then we have the following analogy:

$$\begin{aligned}
\mathbb{Z} &\Longleftrightarrow \mathbb{F}_q[\Sigma^{\mathrm{aff}}],\\
\mathbb{Q} &\Longleftrightarrow \mathbb{F}_q(\Sigma),\\
\mathbb{Z}_p &\Longleftrightarrow \widehat{\mathcal{O}_x} \text{ (completed local ring at } x \in \Sigma),\\
\mathbb{Z}_\infty := \mathbb{R} &\Longleftrightarrow \infty.
\end{aligned}$$

Definition 1.1. An *automorphic function* is a $\mathbb{C}$-valued function on

$$K\backslash \mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\mathbb{Q}),$$

where K being the hyperspecial maximal compact subgroup of $\mathrm{GL}_n(\mathbb{A})$.

For each $x \in \Sigma$, denote F_x the field of fractions of $\widehat{\mathcal{O}_x}$, we have the similar setting for Σ:

$$\prod_{x\in\Sigma} \mathrm{GL}_n(\widehat{\mathcal{O}_x}) \Big\backslash \prod_{x\in\Sigma}{}' \mathrm{GL}_n(F_x) \Big/ \mathrm{GL}_n(F(\Sigma)), \tag{1.1}$$

which is isomorphic to the set $|\mathrm{Bun}_n(\mathbb{F}_q)|$. The proof being easy (manipulating with trivializations of vector bundles, and double quotient corresponds to changing trivializations). Denote the space of automorphic functions associated to Σ by $\mathscr{A}(\Sigma)$.

1.1. Hecke Operators.

We want to construct many commuting operators on the space of automorphic functions. For each $r = 0, \dots, n$, define

$$\begin{aligned}
\mathscr{H}eck^r &= \left\{ (V', V, x) \;\middle|\; \begin{array}{l} V' \text{ isomorphic to a subsheaf of } V, \\ V/V' \text{ is a skyscraper sheaf of rank } r \text{ at } x \end{array} \right\} / \sim \\
&\subset |\mathrm{Bun}_n(\mathbb{F}_q) \times \mathrm{Bun}_n(\mathbb{F}_q) \times \Sigma|.
\end{aligned}$$

Also let the projections from $\mathscr{H}eck^r$ to the isomorphism classes of its three factors $|\mathrm{Bun}_n(\mathbb{F}_q)|$, $|\mathrm{Bun}_n(\mathbb{F}_q)|$, and Σ be $\mathrm{pr}_1, \mathrm{pr}_2, p$ respectively.

We can describe $\mathscr{H}eck^r$ more explicitly. Fix an $x \in \Sigma$, and choose a local uniformizer t at x. Let $V(x)$ be the fiber of V at x, and $E \subset V(x)$ an r-dimensional subspace. Note we have a short exact sequence of $\widehat{\mathcal{O}_x}$-modules

$$0 \to V_x \xrightarrow{t} V_x \to V(x) \to 0.$$

Let $\widetilde{E}$ be the preimage of E in V_x, and we define V' to be the subsheaf of V whose stalks are V_y for any $y \neq x$ and $\widetilde{E}$ at x. Note V' is locally free because Σ is a curve. Therefore, the fiber of $\mathrm{pr}_2 \times p$ over (V, x) in $\mathscr{H}eck^r$ is just the Grassmannian $\mathfrak{Gr}^r(V(x))$.

For any point $x \in \Sigma$, $r = 0, \ldots, n$, define operator

$$\begin{aligned} H_x^r \colon \mathscr{A}(\Sigma) &\to \mathscr{A}(\Sigma) \\ f &\mapsto (\mathrm{pr}_{2,x})_*(\mathrm{pr}_{1,x})^*(f). \end{aligned}$$

Note we regard f as a function on $|\mathrm{Bun}_n(\mathbb{F}_q)|$, and pr_i is viewed here as maps from $\mathscr{H}eck^r$ to $|\mathrm{Bun}_n(\mathbb{F}_q)|$. Then it's easy to see $H_x^r(f)$ at any point (a vector bundle) just sums up the value of f at all subsheaves that have type $(1^r, 0^{n-r})$ at x and are isomorphic elsewhere.

Theorem 1.2. *For any $x, y \in \Sigma$ and any $r, r' = 0, \ldots, n$, the operators H_x^r and $H_y^{r'}$ commute.*

Now we work out some linear algebra for our settings. Let F be a local field with ring of integers $\mathcal{O}$.

Definition 1.3. A lattice in F^n is a finitely generated $\mathcal{O}$-submodule $L \subset F^n$ such that $L \otimes_{\mathcal{O}} F \cong F^n$.

An example of a lattice would be the standard lattice $L_0 = \mathcal{O}^n \subset F^n$.

Lemma 1.4. *Let $L' \subset L$ lattices, then there exists an $\mathcal{O}$-basis $e_1, \ldots, e_n$ of L and integers $m_1 \geq \cdots \geq m_n \geq 0$ such that $L' = t^{m_1}\mathcal{O}e_1 + \cdots + t^{m_n}\mathcal{O}e_n$.*

Proof. This is just the structure theorem for free modules over a PID. □

Corollary 1.5. *Let $L, L' \subset L_0$, then $L \in \mathrm{GL}_n(\mathcal{O})L'$ if and only if $L_0/L \cong L_0/L'$.*

Corollary 1.6. *If $L \subset L_0$, then $\dim_{\mathbb{F}_q} L_0/L = \sum_{i=1}^n m_i$, and if we write $L = gL_0$ for some $g \in \mathrm{GL}_n(F) \cap \mathrm{Mat}_n(\mathcal{O})$, then $\mathrm{val}(\det g) = \dim_{\mathbb{F}_q} L_0/L$.*

All proofs are very easy. Thus we can describe $\mathscr{H}eck^r$ in a slightly different way:

$$\mathscr{H}eck^r = \left\{ (V', V, x) \,\middle|\, \begin{array}{l} \text{there is } V' \to V \text{ injective, and isomorphic on } \Sigma \setminus \{x\}, \\ V'_x \hookrightarrow V_x \text{ has } m_1 = \cdots = m_r = 1 \text{ and } m_{r+1} = \cdots = m_n = 0 \end{array} \right\} / \sim .$$

An observation is that for any pair of lattices L, L', there exists some $m \gg 0$ such that $t^m L' \subset L \subset t^{-m}L'$. Thus we can choose a basis $e_1, \ldots, e_n$ of L such that $L' = t^{m_1}\mathcal{O}e_1 + \cdots + t^{m_n}\mathcal{O}e_n$ for some $m_1 \geq \cdots \geq m_n$ (i.e. dropping the condition $m_1 \geq 0$). We can also find $g \in \mathrm{GL}_n(F)$ such that $L' = gL$ (no longer require $g \in \mathrm{Mat}_n(\mathcal{O})$).

Let affine Grassmannian $\mathfrak{Gr}$ be the set of all lattices in F^n, which is isomorphic to $\mathrm{GL}_n(F)/\mathrm{GL}_n(\mathcal{O})$, by above we see that

$$\mathfrak{Gr} = \coprod_{m_1 \geq \cdots \geq m_n} \mathrm{GL}_n(\mathcal{O}) t^{(m_1, \ldots, m_n)} \mathrm{GL}_n(\mathcal{O})/\mathrm{GL}_n(\mathcal{O}),$$

and equivalently,

$$\mathrm{GL}_n(F) = \coprod_{m_1 \geq \cdots \geq m_n} \mathrm{GL}_n(\mathcal{O}) t^{(m_1, \ldots, m_n)} \mathrm{GL}_n(\mathcal{O}),$$

which is the Cartan decomposition for $GL_n(F)$.

Let $\mathcal{H}$ be the algebra of Hecke operators, $\mathbb{X}_* \cong \mathbb{Z}^n$ be the coroot lattice for GL_n, $\mathbb{C}[\mathbb{X}_*]$ the group ring for $\mathbb{X}_*$, $W \cong \mathfrak{S}_n$ the Weyl group of GL_n, then we have

Theorem 1.7 (Satake)**.**

$$\mathcal{H} \xrightarrow{\sim} \mathbb{C}[\mathbb{X}_*]^W[q^{\pm}] = \mathbb{C}[T^\vee]^W[q^{\pm}] \cong \mathbb{C}[G^\vee]^{G^\vee}[q^{\pm}],$$

with basis of characters $\chi_{(m_1,\dots,m_n)}$.

2. March 29

Let $\Lambda = \mathbb{Z}^n$ with $\mathfrak{S}_n$ acting by permutation. Let $G = \mathrm{GL}_n$, k be any field, $F = k((t))$ the field of Laurent series, and $\mathcal{O} = k[[t]]$ the ring of power series. We have defined the affine Grassmannian $\mathfrak{Gr}$ to be the set of all rank n $\mathcal{O}$-lattices in F^n. The group $G(F)$ acts transitively on $\mathfrak{Gr}$ and the stablizer of the standard lattice $L_0 = \mathcal{O}^n$ is $G(\mathcal{O})$. Therefore $\mathfrak{Gr} = G(F)/G(\mathcal{O})$.

Definition 2.1. Let (L, L') be a pair of lattices. A *basis adapted to* (L, L') is such $v_1, \dots, v_n \in L$ that $L = \mathcal{O}v_1 + \dots + \mathcal{O}v_n$ and $L' = t^{m_1}\mathcal{O}v_1 + \dots + t^{m_n}\mathcal{O}v_n$. The adapted basis is said to have *type* $\lambda = (m_1, \dots, m_n)$.

Lemma 2.2.
1. *Any pair* (L, L') *has an adapted basis, and its type* λ *is unique up to permutations. In this case we say* (L, L') *are in relative position* $\lambda \mod \mathfrak{S}_n \in \Lambda/\mathfrak{S}_n$.
2. *Two pairs* (L_1, L_2) *and* (L_1', L_2') *are in the same relative position if and only if they belong to the same orbit of the action of* $G(F)$ *in* $\mathfrak{Gr} \times \mathfrak{Gr}$.

Remark 2.3. Note that for any two groups $B \subset A$, we have natural bijection

$$\begin{aligned} A\backslash(A/B \times A/B) &\to B\backslash A/B \\ (a, a') &\mapsto a^{-1}a'. \end{aligned}$$

Therefore $G(F)\backslash(\mathfrak{Gr} \times \mathfrak{Gr}) \cong G(\mathcal{O})\backslash\mathfrak{Gr} \cong G(\mathcal{O})\backslash G(F)/G(\mathcal{O})$.

Proof of Lemma 2.2. For the first part, note there exists $r \gg 0$ such that $t^r L' \subset L$, then by Lemma 1.4 we can find an adapted basis for $(L, t^r L')$. Dividing by t^r on the coefficients we get an adapted basis for (L, L'). The uniqueness can also be seen easily from Lemma 1.4.

For the second part, let $v_1, \dots, v_n$ be a basis adapted to (L_1, L_2), and $v_1', \dots, v_n'$ one to (L_1', L_2'). Assume they have the same relative position, then by permuting v_i' (possible by acting by an element in $G(F)$) we can assume they have the same type. Then we can define $g \in G(F)$ to be the element sending v_i to v_i', then $g(L_1) = L_1'$. The same type assumption says that $g(L_2) = L_2'$ as well. The other direction is proved by running the argument backwards, which is more trivial. □

Since $\mathfrak{Gr}$ is not of finite type, we now write it as a limit of (projective) varieties. Fix an integer $r > 0$, we certainly have $t^r L_0 \subset t^{-r} L_0$. Define

$$\mathfrak{Gr}_r = \{L \in \mathfrak{Gr} \mid t^r L_0 \subset L \subset t^{-r} L_0\},$$

we then have $\mathfrak{Gr} = \varinjlim \mathfrak{Gr}_r$. We can rewrite it as

$$\mathfrak{Gr}_r \cong \{L/t^r L_0 \subset t^{-r} L_0/t^r L_0 \cong (k^n)^{2r} \mid L/t^r L_0 \text{ is } t\text{-stable}\},$$

where the uniformizer t acts nilpotently on $(k^n)^{2r}$.

Remark 2.4.
1. When $k = \mathbb{F}_q$, $\mathfrak{Gr}_r$ is a finite set.
2. When $k = \mathbb{C}$, $\mathfrak{Gr}_r$ is a projective variety.

(3) The action of t on $\mathfrak{Gr}_r$ can be described by coordinates. Choose the standard $\mathcal{O}$-basis $e_1, \dots, e_n$ for L_0, then a k-basis for $t^{-r}L_0/t^rL_0$ is t^je_i ($1 \le i \le n, -r \le j < r$). Under this basis, the action of t can be identified with the $2nr \times 2nr$-matrix

$$\begin{pmatrix} J_0 & 0 & \cdots & 0 \\ 0 & J_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_0 \end{pmatrix},$$

where J_0 is the Jordan block of size $2r$ with eigenvalue 0.

Denote $T \subset G$ the subgroup of invertible diagonal matrices, and N the subgroup of uppertriangular unipotent matrices.

Lemma 2.5. (1) *There is a bijection $\Lambda \cong \mathfrak{Gr}^{T(k)}$ identifying $\lambda = (m_1, \dots, m_n)$ with $L^\lambda = t^{m_1}\mathcal{O}e_1 + \cdots + t^{m_n}\mathcal{O}e_n$, where e_i is the standard basis for L_0.*

(2) *We have*

$$\mathfrak{Gr} = \coprod_{\lambda \in \Lambda} N(F)L^\lambda,$$

or equivalently,

$$G(F) = \coprod_{\lambda \in \Lambda} N(F)t^\lambda G(\mathcal{O}).$$

The latter is called Iwasawa decomposition*.*

Proof. For the first part, first note that the map $\lambda \mapsto L^\lambda$ is clearly injective. For surjectivity, note that the action of $T(k)$ on $\mathfrak{Gr}$ induces a $T(k)$-action on $V = t^{-r}L_0/t^rL_0$, which commutes with the action of t. It is clear that any $T(k)$-stable subspace $E \subset V$ has a form $E = E_1 \oplus \cdots \oplus E_n$ where $E_i \subset \bigoplus_{j=-r}^{r-1} k(t^je_i)$. If E is in addition t-stable, then $E_i = \bigoplus_{j=m_i}^{r-1} k(t^je_i)$ for some $m_i \ge -r$.

For the second part, note any lattice can be transferred through Gaussian elimination with $\mathcal{O}$-coefficients to some L^λ. $\square$

Now we describe the Hecke operators for curves in a different way. Let Σ be a smooth projective curve over k, and fix a k-point $x \in \Sigma$. Let $\mathcal{O} = \widehat{\mathcal{O}_x}$ be the completed local ring at x, and F the fraction field of $\mathcal{O}$. For convenience, let $\mathcal{O}_{\text{out}} = \mathcal{O}_{\Sigma - \{x\}}$. By Bun_n we mean the stack of vector bundles of rank n on Σ. We claim that

$$\mathfrak{Gr} \cong \{(V, \psi) \mid V \in \text{Bun}_n(\Sigma), \psi \text{ a trivialization of } V \text{ on } \Sigma - \{x\}\}/\sim.$$

The proof of the claim is straightforward: recall in last lecture we have that the double quotient (1.1) is isomorphic to $|\text{Bun}_n(k)|$, and those with a trivialization over $\Sigma - \{x\}$ correspond to the double coset represented by those with only one factor (i.e. at x) not in the local integral points. Now we put back the trivialization outside of x, we get the result.

For any $\lambda \in \Lambda/\mathfrak{S}_n$, we define

$$\mathscr{H}eck_x^\lambda = \left\{ (V, V', \varphi) \;\middle|\; \begin{array}{l} \varphi : V|_{\Sigma-\{x\}} \xrightarrow{\sim} V'|_{\Sigma-\{x\}} \\ (V_x, V'_x) \text{ in relative position } \lambda \text{ in } F \otimes_\mathcal{O} V_x \xrightarrow{\sim} F \otimes_\mathcal{O} V'_x \text{ via } \varphi \end{array} \right\} / \sim.$$

Denote still by pr_1, pr_2 projections to V and V' respectively. Let Bun_n^0 be the substack of bundles that are trivial on $\Sigma - \{x\}$. Therefore by above discussion we have

$$\begin{aligned}\mathrm{pr}_1^{-1}(\mathrm{Bun}_n^0) &= G(\mathcal{O}_{\mathrm{out}})\backslash(\mathfrak{Gr}\times\mathfrak{Gr})^\lambda \\ &= G(\mathcal{O}_{\mathrm{out}})\backslash G(F)/\operatorname{Stab}(L^0, L^\lambda),\end{aligned}$$

where the superscript λ denotes the pairs of relative position λ.

In order to describe the alternative description for Hecke algebra, we first state some results in a more general setting. Let M be a locally compact topological group, and K a maximal compact subgroup. Choose a Haar measure μ such that $\mu(K) = 1$. Let $C_{\mathrm{c}}(K\backslash M/K)$ be the $\mathbb{C}$-valued K-biinvariant continuous functions with compact support on M. It is an algebra under convolution. It can be shown to be naturally isomorphic to $C_{\mathrm{c}}(M\backslash(M/K\times M/K))$ which also has a natural convolution (with the support conditions for the functions properly defined).

Remark 2.6. For a space X, and good enough functions f, g on X. The convolution can be defined by $(f, g) \mapsto p_{13,*}(p_{12}^* f \cdot p_{23}^* g)$, where p_{ij} are the three projections from $X\times X\times X$ to $X \times X$.

When Y is a space with M acting on it, we have a natural action of $C_{\mathrm{c}}(K\backslash M/K)$ on $C_{\mathrm{c}}(Y/K)$:

$$\begin{aligned}C_{\mathrm{c}}(Y/K)\otimes C_{\mathrm{c}}(K\backslash M/K) &\to C_{\mathrm{c}}(Y/K) \\ f\otimes g &\mapsto (y \mapsto \int_M f(ym)g(m^{-1})\mathrm{d}m).\end{aligned}$$

Now when $Y = \{(V, v) \mid V \in \mathrm{Bun}_n, v \text{ is an } \mathcal{O} \text{ basis of } V_x\}$, $M = G(F)$, $K = G(\mathcal{O})$, we have $C_{\mathrm{c}}(G(\mathcal{O})\backslash G(F)/G(\mathcal{O}))$ acting on $C_{\mathrm{c}}(Y/K)$.

Theorem 2.7. *The algebra $C_{\mathrm{c}}(G(\mathcal{O})\backslash G(F)/G(\mathcal{O}))$ is commutative.*

Proof by Gelfand. For general $K \subset M$, suppose we have an anti-involution τ on M such that $\tau(KmK) = KmK$ for all $m \in M$. We claim $C_{\mathrm{c}}(K\backslash M/K)$ is commutative: define τ acting on $C_{\mathrm{c}}(M)$ by $(\tau f)(m) = f(\tau(m))$. Then it is easily verified $\tau(f * g) = \tau(g) * \tau(f)$. On the other hand, by assumption on τ, we know $\tau(f) = f$ for all $f \in C_{\mathrm{c}}(K\backslash M/K)$, we have $\tau(f * g) = f * g$ and $\tau(f * g) = g * f$, so $C_{\mathrm{c}}(K\backslash M/K)$ is commutative. Going back to our specific case, choose τ to be the transposition of matrices which, by Cartan decomposition, satisfies the assumption on the anti-involution above, so we are done. ∎

3. April 3

In this lecture we establish some facts about affine Grassmannian for a general reductive (actually semisimple only) group, analogous to the GL_n case (which is not semisimple hence it's not a perfect analogy). Let k be an arbitrary field, $F = k((\varpi))$ the field of Laurent series over k, $\mathcal{O} = k[[\varpi]]$ the ring of formal power series. Let G be a split reductive group over k, and we define

$$\mathfrak{Gr}_G := G(F)/G(\mathcal{O}).$$

Let $\mathfrak{g} = \operatorname{Lie} G$, and fix a nondegenerate invariant symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$, i.e. for all $x, y, z \in \mathfrak{g}$, we have $([x, y], z) + (y, [x, z]) = 0$. We can extend it to a bilinear form on

$\mathfrak{g}(F) \times \mathfrak{g}(F) \to F$. For example, when $\mathfrak{g} = \mathfrak{sl}_n$, we can choose the Killing form $(x, y) = \mathrm{Tr}(\operatorname{ad} x \circ \operatorname{ad} y)$, and when G is reductive, we can extend the Killing form from its semisimple part by putting a nondegenerate symmetric bilinear form (automatically invariant) on its abelian part, and demanding the two parts orthogonal to each other.

Now assume G is semisimple.

Definition 3.1. A *lattice* in $\mathfrak{g}(F)$ is a finitely generated $\mathcal{O}$-submodule $L \subset \mathfrak{g}(F)$ such that $L \otimes_{\mathcal{O}} F \cong \mathfrak{g}(F)$. The *dual lattice* $L^\vee := \mathrm{Hom}_{\mathcal{O}}(L, \mathcal{O})$, identified as a lattice in $\mathfrak{g}(F)$ as well through the bilinear form $(\cdot, \cdot)$.

Then we can give another definition of affine Grassmannian:

$$\mathfrak{Gr}'_G := \{L \subset \mathfrak{g}(F) \text{ lattice} \mid [L, L] \subset L, \text{ and } L = L^\vee\}.$$

We want to connect $\mathfrak{Gr}_G$ with $\mathfrak{Gr}'_G$ (i.e. to show they are isomorphic). As usual, we introduce the standard lattice $L_0 = \mathfrak{g}(\mathcal{O})$, and let $G(F)$ act on $\mathfrak{Gr}'_G$ by the adjoint action: $g \colon L \mapsto \operatorname{Ad} g(L)$. Note that $\mathrm{Stab}_{G(F)} L_0 = G(\mathcal{O})$, and the action of $G(F)$ is transitive. Thus we get $\mathfrak{Gr}'_G \cong G(F)/G(\mathcal{O}) = \mathfrak{Gr}_G$.

Remark 3.2. The two definitions can be seen in methodology parallel to the definitions of flag varieties. The definition of $\mathfrak{Gr}_G$ is similar to defining the flag variety by G/B with some chosen Borel B, and that of $\mathfrak{Gr}'_G$ is choice-free, thus similar to defining the flag variety by the functor of points (set of all Borel subgroups in G).

Lemma 3.3. *For all lattice $L \subset \mathfrak{g}(F)$, there exists $n \gg 0$ such that $\varpi^n L_0 \subset L \subset \varpi^{-n} L_0$. In other words,*

$$\frac{L}{\varpi^n L_0} \subset \frac{\varpi^{-n} L_0}{\varpi^n L_0} =: V_n.$$

Define a bilinear form

$$\begin{aligned} \beta_n \colon V_n \times V_n &\to k \\ (x, y) &\mapsto \mathrm{Res}_{\varpi=0}(x, y), \end{aligned}$$

where the parentheses on the right hand side denote the bilinear form on $\mathfrak{g}(F)$, and $\mathrm{Res}_{\varpi=0}$ simply means taking the coefficient of ϖ^{-1}. It is easy to check that β_n is nondegenerate, and we can describe $\mathfrak{Gr}_G$ as a limit of projective varieties:

$$\mathfrak{Gr}_n := \{V \subset V_n \mid V \text{ is } \varpi\text{-stable, maximal isotropic w.r.t. } \beta_n, \text{ and the 3-form } \beta_n([\cdot,\cdot],\cdot) = 0 \text{ on } V\},$$

and $\mathfrak{Gr}_n$ comes with a natural structure of a projective variety over k. It is clear that we have the following isomorphism

$$\begin{aligned} \{L \in \mathfrak{Gr}_G \mid \varpi^n L_0 \subset L \subset \varpi^{-n} L_0\} &\xrightarrow{\sim} \mathfrak{Gr}_n \\ L &\mapsto \frac{L}{\varpi^n L_0}, \end{aligned}$$

with the maximal isotropic condition on the right corresponding to the self-dual condition on the left, and after assuming this, the Lie subalgebra condition on the left is equivalent to the 3-form condition on the right (which instead of putting conditions on L only, translating some of those conditions to the dual via β_n).

Corollary 3.4. *We have* $\mathfrak{Gr}_G = \varinjlim_n \mathfrak{Gr}_n$.

The $G(\mathcal{O})$-action on lattice by conjugation descends to an action on V_n. Let $T \subset G$ be a split maximal torus, and let $\lambda \in \mathbb{X}_* = \mathbb{X}_*(T) = \mathrm{Hom}_k(\mathbb{G}_\mathrm{m}, T) \cong \mathbb{Z}^{\mathrm{rk}\, G}$. We have an element $\varpi^\lambda \in T(F)$. Similar as before, we let $L^\lambda = \mathrm{Ad}\,\varpi^\lambda(L_0)$. Let $\mathfrak{t} = \mathrm{Lie}\, T$, $\Phi \subset \mathfrak{t}^*$ the root system of $(\mathfrak{g}, \mathfrak{t})$, and (by choosing a Borel) fix a set of positive roots. Thus we have the decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} k e_\alpha,$$

where e_α is a nonzero vector in $\mathfrak{g}$. Therefore we have

$$\mathfrak{g}(\mathcal{O}) = \mathfrak{t}(\mathcal{O}) \oplus \bigoplus_{\alpha\in\Phi} \mathcal{O} e_\alpha,$$

and so

$$\mathrm{Ad}\,\varpi^\lambda \mathfrak{g}(\mathcal{O}) = \mathfrak{t}(\mathcal{O}) \oplus \bigoplus_{\alpha\in\Phi} \varpi^{\langle\lambda,\alpha\rangle} \mathcal{O} e_\alpha,$$

Assume λ is dominant, then if $\langle\lambda,\alpha\rangle \le n$ for all $\alpha > 0$, then we get $\varpi^n L_0 \subset L^\lambda$, and automatically $L^\lambda \subset \varpi^{-n} L_0$ by self-duality.

We define $\mathfrak{Gr}^\lambda$ to be the $G(\mathcal{O})$-orbit of L^λ, then $\mathfrak{Gr}^\lambda = G(\mathcal{O})/\operatorname{Stab}_{G(\mathcal{O})} L^\lambda$. We can describe the Lie algebra of the stablizer explicitly if λ is dominant and $\langle\lambda,\alpha\rangle \le n$: since $\operatorname{Stab}_{G(\mathcal{O})} L^\lambda = G(\mathcal{O}) \cap \operatorname{Stab}_{G(F)} L^\lambda = G(\mathcal{O}) \cap \mathrm{Ad}\,\varpi^\lambda(G(\mathcal{O}))$,

$$\begin{aligned}\mathrm{Lie}\,(G(\mathcal{O}) \cap \mathrm{Ad}\,\varpi^\lambda(G(\mathcal{O}))) &= \mathfrak{g}(\mathcal{O}) \cap \mathrm{Ad}\,\varpi^\lambda(\mathfrak{g}(\mathcal{O})) \\ &= \mathfrak{t}(\mathcal{O}) \oplus \bigoplus_{\alpha\in\Phi^+} \varpi^{\langle\lambda,\alpha\rangle}\mathcal{O} e_\alpha \oplus \bigoplus_{\alpha\in\Phi^-} \mathcal{O} e_\alpha.\end{aligned}$$

Thus we can calculate the dimension of $\mathfrak{Gr}^\lambda$:

$$\dim_k \mathfrak{Gr}^\lambda = \dim_k \frac{\mathfrak{g}(\mathcal{O})}{\mathrm{Lie}(\operatorname{Stab}_{G(\mathcal{O})} L^\lambda)} = \sum_{\alpha>0} \langle\lambda,\alpha\rangle = 2\langle\lambda,\rho\rangle,$$

where ρ is one half of the sum of all positive roots.

Lie theory tells us that the coroot lattice $Q^\vee \subset \mathbb{X}_*$ as a finite index sublattice. We also know from Lie theory that $\langle\lambda,\rho\rangle \in \mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}$ if $\lambda \in Q^\vee$ or $\mathbb{X}_*$ respectively.

Corollary 3.5. *The orbit* $\mathfrak{Gr}^\lambda$ *is even-dimensional if* $\lambda \in Q^\vee$.

Now let $k = \mathbb{C}$. Choose a maximal compact subgroup $K \subset G$, and define the (polynomial) loop space of K

$$\Omega(K) := \{\text{polynomial maps } f\colon S^1 \to K \text{ such that } f(1) = 1\}.$$

We can view G as embedded in $\mathrm{GL}_n(\mathbb{C})$ for some n, and polynomial maps from $\mathbb{C}$ to G are simply polynomial maps in coordinates. A polynomial map to K is one that is the restriction (to S^1) of some polynomial map $\mathbb{C} \to G$ whose image of S^1 lies in K. We then have maps

$$\Omega(K) \hookrightarrow G(\mathbb{C}((\varpi))) \to \mathfrak{Gr}_G = G(\mathbb{C}((\varpi)))/G(\mathbb{C}[[\varpi]]).$$

Theorem 3.6. *The composition of the above map* $\Omega(K) \to \mathfrak{Gr}_G$ *is an isomorphism.*

The proof for GL_n is essentially the Gram-Schmidt process.

Lastly, for a chosen (G, K), we have the simply-connected as well as the adjoint type isogenus groups K^{sc} and K^{ad} for K, and the maps

$$K^{\mathrm{sc}} \to K \to K^{\mathrm{ad}}$$

being both group quotients and covering maps. Clearly $\pi_1(K^{\mathrm{ad}}) = Z(K^{\mathrm{sc}})$. So if K is the simply-connected type, we have

$$\pi_0(\Omega(K^{\mathrm{sc}})) \cong \pi_1(K^{\mathrm{sc}}) = 1,$$

and $\mathfrak{Gr}_{G^{\mathrm{sc}}}$ is connected. If K is the adjoint type, then

$$\pi_0(\Omega(K^{\mathrm{ad}})) \cong \pi_1(K^{\mathrm{ad}}) = Z(K^{\mathrm{sc}}),$$

and moreover we have a natural map from $\mathfrak{Gr}_{G^{\mathrm{sc}}}$ to $\mathfrak{Gr}_{G^{\mathrm{ad}}}$ realizing the former as a connected component of the latter.

4. April 5

This lecture will define the Grassmannian as a scheme (more precisely an ind-scheme). We deal with the case $k = \mathbb{C}$ first, and mention a bit about the mixed characteristic case. As usual $F = k((\varpi))$ and $\mathcal{O} = k[[\varpi]]$.

Let H be a linear algebraic group, then we can define the functor of arc space of H

$$\begin{aligned} \underline{H(\mathcal{O})} \colon \mathrm{Alg}/k &\to \mathrm{Sets} \\ R &\mapsto H(R[[\varpi]]). \end{aligned}$$

Proposition 4.1. *This functor is representable by a scheme of infinite type $H(\mathcal{O})$.*

Similarly, we can define the functor of loop space of H

$$\begin{aligned} \underline{H(F)} \colon \mathrm{Alg}/k &\to \mathrm{Sets} \\ R &\mapsto H(R((\varpi))). \end{aligned}$$

Proposition 4.2. *This functor is representable by an ind-scheme $H(F)$, i.e. it is a direct limit (as a functor) of closed embeddings of schemes.*

Definition 4.3. The affine Grassmannian $\mathfrak{Gr}_G$ is defined to be the quotient of functors (as fpqc sheaves of groups) $G(F)/G(\mathcal{O})$.

Remark 4.4.
1. The ind-scheme $G(F)$ is in general not reduced, even for the simplest groups like $\mathbb{G}_{\mathrm{m}}$;
2. The quotient map $G(F) \to \mathfrak{Gr}_G$ is a $G(\mathcal{O})$-torsor in the fpqc topology.

4.1. GL_n-Case.

Let $G = \mathrm{GL}_n$, and R a k-algebra.

Definition 4.5. A *lattice* is a finitely generated projective $R[[\varpi]]$-module $L \subset R((\varpi))^n$ such that $R((\varpi)) \otimes_{R[[\varpi]]} L \cong R((\varpi))^n$.

As usual we have the standard lattice $L_0 = R[[\varpi]]^n$, and for all lattice L, there exists a large m such that $\varpi^m L_0 \subset L \subset \varpi^{-m} L_0$. To better connect the lattices to the geometric description of Grassmannians (i.e. through vector bundles), so that we can get vector bundles not just coherent sheaves, we need this lemma:

Lemma 4.6. *The quotient $\varpi^{-m} L_0 / L$ is a projective R-module.*

4.2. **Case of mixed characteristics.** In this subsection only, we talk briefly about the number-theoretic settings. Let $k = \mathbb{F}_p$, $\mathcal{O} = \mathbb{Z}_p$, and $F = \mathbb{Q}_p$. We want to define an ind-scheme $\mathfrak{Gr}$ over k such that it represents the functor of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p$.

The first step is to define the analogue of lattices. Note that $\mathbb{Z}_p$ can be seen as the ring of Witt vectors over k, denoted by $W(k)$, and $\mathbb{Q}_p = W(k)[p^{-1}]$. Let R be a k-algebra, it is tempting to define the R-points of the functor to be finitely generated projective $W(R)$-modules inside $W(R)[p^{-1}]^n$ of generic rank n. However, if R is non-reduced, then p may be a zero divisor in $W(R)$, thus $W(R)$ is not a subring of $W(R)[p^{-1}]$.

To remedy this, we can restrict the domain category to all *perfect* k-algebras, i.e. those R such that the Frobenius $x \mapsto x^p$ is bijective.

Lemma 4.7. *If R is perfect then p is not a zero divisor in $W(R)$.*

Now consider the functor

$$\begin{aligned} \mathrm{PerfAlg}/k &\to \mathrm{Sets} \\ R &\mapsto \text{lattices in } W(R)[p^{-1}]^n, \end{aligned}$$

then we have:

Proposition 4.8. *This functor is representable by an ind-scheme $\varinjlim_i X_i$ with each X_i a perfect scheme (affine-locally being the spectra of perfect k-algebras).*

However, another problem arises since perfect k-algebras are in general not of finite type, because usually it is a perfect closure of some k-algebra by taking p-th root over and over again. Only until recently have people understood a deeper result:

Theorem 4.9 (Bhatt-Scholze, 2015)**.** *With the notations above, X_i is a perfection of a projective variety over k, and the embeddings $X_i \hookrightarrow X_{i+1}$ come from the maps of those varieties.*

However, for a general reductive group not much is known yet.

4.3. **General reductive group case.** For the remaining of this section $k = \mathbb{C}$. We start with the tori. Let $G = T \cong {\mathbb{G}_\mathrm{m}}^n$, and $X_* = X_*(T) = \mathrm{Hom}(\mathbb{G}_\mathrm{m}, T)$, $X^* = X^*(T) = \mathrm{Hom}(T, \mathbb{G}_\mathrm{m})$ be the coweight and weight lattices respectively. We have a natural isomorphism $X^* \cong \mathrm{Hom}_\mathbb{Z}(X_*, \mathbb{Z})$, since there exists a perfect pairing

$$\begin{aligned} X^* \times X_* &\to \mathbb{Z} \\ (\chi, \gamma) &\mapsto n, \end{aligned}$$

where $\chi \circ \gamma = (z \mapsto z^n)$. One can also recover T from the lattices by noticing $T \cong \mathbb{G}_\mathrm{m} \otimes_\mathbb{Z} X_*$. This suggests the definition $T^\vee = \mathbb{G}_\mathrm{m} \otimes_\mathbb{Z} X^*$, and it is clear that $X_*(T^\vee) = X^*$ and $X^*(T^\vee) = X_*$. One can also construct $T^\vee$ directly from T in this case since topologically, $X_* \cong \pi_1(T)$, and so $T^\vee \cong \mathrm{Hom}(\pi_1(T), \mathbb{G}_\mathrm{m})$, which is the same as rank-1 local systems on T.

It turns out we can largely ignore the non-reducedness of the Grassmannian, thus we will focus on the underlying space of points. Since $F^\times/\mathcal{O}^\times \cong \mathbb{Z}$ through valuation, we have as a space $\mathfrak{Gr}_T = T(F)/T[[\varpi]] \cong X_*(T)$.

Next we consider $G = \mathrm{SL}_n \subset \mathrm{GL}_n$. We claim that $\mathfrak{Gr}_{\mathrm{SL}_n} \subset \mathfrak{Gr}_{\mathrm{GL}_n}$. Indeed, we can identify the former with the lattices in F^n satisfying some additional conditions.

Definition 4.10. For two lattices $L, L' \subset F^n$, we define the *relative length*

$$\mathrm{Leng}(L, L') = \dim_{\mathbb{C}}(L/(L \cap L')) - \dim_{\mathbb{C}}(L'/(L \cap L')),$$

necessarily a finite number.

The following is straightforward.

Lemma 4.11. *We have the isomorphism (as ind-schemes)* $\mathfrak{Gr}_{\mathrm{SL}_n} \cong \{L \in \mathfrak{Gr}_{\mathrm{GL}_n} \mid \mathrm{Leng}(L, L_0) = 0\}$.

We define the determinant bundle on $\mathfrak{Gr}_{\mathrm{SL}_n}$ by letting the fiber over L be $\det(L/(L \cap L_0)) \otimes_{\mathbb{C}} \det(L^0/(L \cap L_0))^\vee$.

Proposition 4.12. *The determinant bundle is an ample line bundle on* $\mathfrak{Gr}_{\mathrm{SL}_n}$ *hence gives a projective embedding of* $\mathfrak{Gr}_{\mathrm{SL}_n}$ *(as ind-schemes).*

The determinant bundle can be generalized to any reductive group G. The loop Lie algebra $\mathfrak{g}(F)$ admits an important central extension

$$0 \to \mathbb{C} \to \widehat{\mathfrak{g}} \to \mathfrak{g}(F) \to 0,$$

where $\widehat{\mathfrak{g}}$ is a Kac-Moody algebra. The construction is explicit: fix a symmetric nondegenerate invariant bilinear form $\beta\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, the Kac-Moody algebra is defined as a vector space simply $\widehat{g} = \mathbb{C} \oplus \mathfrak{g}(F)$, on which the cocycle $c_\beta \in \mathrm{H}^2(\mathfrak{g}(F), \mathbb{C})$ for the central extension is given by $c_\beta(x, y) = \mathrm{Res}_{\varpi=0}\, \beta(x, y)$. The same can be done for groups as well, i.e. we have a central extension

$$1 \to \mathbb{G}_{\mathrm{m}} \to \widehat{G} \to G(F) \to 1,$$

in which the preimage of $G(\mathcal{O})$ splits. Therefore we have a $\mathbb{G}_{\mathrm{m}}$-torsor

$$\widehat{G}/G(\mathcal{O}) \to G(F)/G(\mathcal{O}) = \mathfrak{Gr}_G.$$

The associated line bundle is the determinant bundle det. For GL_n or SL_n, the bilinear form β can be chosen as the Killing form (extended to GL_n as in Section 3).

For any simply-connected G, let $\mathbb{V}_\beta = \mathrm{Ind}_{\mathfrak{g}(\mathcal{O})}^{\widehat{\mathfrak{g}}} 1_c$, for which the central character c is some fixed one, we then have the identification (verify it)

$$\widehat{G} = \mathbb{P}(\mathbb{V}_\beta^*).$$

4.4. **Another example:** PGL_n**.** Let $G = \mathrm{PGL}_n = \mathrm{GL}_n/\mathbb{G}_{\mathrm{m}}$. It's not hard to see

$$\mathfrak{Gr}_{\mathrm{PGL}_n} \cong \mathfrak{Gr}_{\mathrm{GL}_n}/(L \sim \varpi L).$$

With this identification, any element in $\mathfrak{Gr}_{\mathrm{PGL}_n}$ is then represented by some lattice of relative length $r \in \{0, \dots, n-1\}$ to L_0, and any two elements are in the same connected component if and only if their relative lengths to L_0 are the same ($\mod n$). For each $m = 0, \dots, n-1$, let

$$\begin{aligned}\mathfrak{Gr}^m &= \{L \subset L_0 \mid L \text{ is the preimage of some } m\text{-dimensional subspace of } L_0/\varpi L_0\} \\ &\cong \mathfrak{Gr}_m(\mathbb{C}^n),\end{aligned}$$

the last term being the usual Grassmannian in $\mathbb{C}^n$.

Proposition 4.13. $\mathfrak{Gr}_{\mathrm{PGL}_n}$ *has n connected components, and* $\mathfrak{Gr}^m$ *is the unique closed $G(\mathcal{O})$-orbit in the connected component corresponding to m.*

4.5. **Cartan and Iwasawa decompositions.** This subsection is probably going to be repeated in the next lecture. For each $\lambda \in X_*(T)$, we have $\varpi^\lambda \in T(F) \in G(F)$, hence $\varpi^\lambda G(\mathcal{O})/G(\mathcal{O}) \subset \mathfrak{Gr}_G$. Let N be a maximal unipotent subgroup.

Theorem 4.14 (Cartan Decomposition)**.**

$$\mathfrak{Gr}_G = \bigcup_{\lambda \in X_*(T)} G(\mathcal{O})\varpi^\lambda.$$

Theorem 4.15 (Iwasawa Decomposition)**.**

$$\mathfrak{Gr}_G = \bigcup_{\lambda \in X_*(T)} N(F)\varpi^\lambda.$$

To prove these we need a lemma.

Lemma 4.16.

$$(\mathfrak{Gr}_G)^T \cong \{\varpi^\lambda \mid \lambda \in X_*\}.$$

Sketch of the Proof (to be finished next lecture). Choose a sufficiently general one-parameter subgroup $\gamma\colon \mathbb{G}_{\mathrm{m}} \to T$ such that $(\mathfrak{Gr}_G)^\gamma = (\mathfrak{Gr}_G)^T$. Any $G(\mathcal{O})$-orbit X in $\mathfrak{Gr}_G$ is a projective variety (why?), and is γ-stable. Fix a point $x \in X$, and look at the γ-orbit of x, we have an algebraic function $\mathbb{C}^\times \to X$. But X is projective, so this map extends to $\mathbb{C} \to X$. Let $x_0 = \lim_{z\to 0} \gamma(z) \cdot x$, then $x_0 \in (\mathfrak{Gr}_G)^T \cap X$, which proves Theorem 4.14. □