# NOTES ON $p$-ADIC INTEGRALS

GRIFFIN WANG

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This note explains basics of $p$-adic integrals used in an algebra setting. Throughout this note, $F$ is a non-archimedean local field and $\mathcal{O}$ is its ring of integers with residue field $k$. Let $p = \mathrm{char}(k)$ and $q = \#k$.

## 1. $\mathcal{O}$-varieties

We start with the definition of an $\mathcal{O}$-variety.

**Definition 1.1.** An $\mathcal{O}$-variety is a finite type $\mathcal{O}$-scheme that is reduced, separated, and flat over $\mathcal{O}$.

*Remark* 1.2. In equal characteristic case, for any $k$-variety $X_0$, one can base change to get an $\mathcal{O}$-variety $X = X_0 \times_{\operatorname{Spec} k} \operatorname{Spec} \mathcal{O}$. In general, one should start with $\mathcal{O}$-varieties.

There is a specialization map $\odot\colon X(\mathcal{O}) \to X(k)$. On the other hand, since $X$ is separated, we have natural inclusion $X(\mathcal{O}) \subset X(F)$. If $X$ is also smooth over $\mathcal{O}$, then $X(\mathcal{O})$ is naturally a compact open analytic $F$-submanifold of $X(F)$.

## 2. The Weil Measure and Point Counting

Let $X$ be a smooth $\mathcal{O}$-variety of relative dimension $n$. There is a canonical (real-valued) measure, i.e. the Weil measure, on $X(\mathcal{O})$, constructed as follows. Let $\omega_{X/\mathcal{O}}$ be the relative canonical sheaf on $X$. Choose an affine open covering $\{U_i\}$ of $X$ of $\mathcal{O}$-schemes, over which $\omega_{X/\mathcal{O}}$ is trivialized, and fix a trivialization.

**Lemma 2.1.** *We have $X(\mathcal{O}) = \bigcup_i U_i(\mathcal{O})$.*

*Proof.* Consider $x\colon \operatorname{Spec} \mathcal{O} \to X$, then the special point $x_1$ of $x$ is sent to one of $U_i$. Then the generic point $x_0$ of $x$ is in $U_i$ as well, otherwise $x_0 \in X \setminus U_i$, and since $X \setminus U_i$ is closed, $x_1 \in \overline{\{x_0\}} \subset X \setminus U_i$, which is a contradiction. ▲

Over each $U_i$, choose a generator $s_i$ of $\omega_{X/\mathcal{O}}|_{U_i}$, then $s_i$ is nowhere vanishing on $U_i$. In local coordinates, $s_i = f_i \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, and the measure on $U_i$ is defined by $|f_i|_F \mathrm{d}x_1 \cdots \mathrm{d}x_n$, where $\mathrm{d}x_1 \cdots \mathrm{d}x_n$ denotes the Haar measure on $F^n$, normalized so that $\mathcal{O}^n$ has measure 1. Just like in calculus, one can prove an analogue of change-of-variable formula for integration on $F^n$, and thus show this measure is independent of the local coordinates.

Suppose we have another generator $s_i'$ of $\omega_{X/\mathcal{O}}|_{U_i}$, then it differs from $s_i$ by some invertible function $g_i\colon U_i \to \mathbb{G}_\mathrm{m} = \operatorname{Spec} \mathbb{Z}[T^{\pm}]$. For each $x \in U_i(\mathcal{O})$, we get an associated map $g_i(x)^{\#}\colon \mathbb{Z}[T^{\pm}] \to \mathcal{O}$, and the value $g_i(x) = (g_i(x)^{\#})(T) \in \mathcal{O}^\times$. Therefore $|g_i|_F = 1$ everywhere on $U_i(\mathcal{O})$, and this shows the measure is independent of the choice of $s_i$. For the same reason, the measures on different charts glue on the overlap, giving us a measure on $X(\mathcal{O})$.

*Remark* 2.2. As [Yas14] shows, one can replace $\omega_{X/\mathcal{O}}$ by something slightly more general. See also the discussion in section 5.

Let $Z \subset X(k)$ be any subset, and $\mathcal{D}_X(Z) = \odot^{-1}(Z)$ be the preimage of $Z$ under the specialization map. This is an *open* subset of $X(\mathcal{O})$. The following results due to Weil are well-known, and the proofs are straightforward.

**Theorem 2.3.** *One has $\mu(\mathcal{D}_X(Z)) = q^{-n}\#Z$, where $\mu$ is the Weil measure on $X$.*

**Corollary 2.4.** *Let $f\colon X(k) \to \mathbb{C}$ be any function. We have*

$$\int_{X(\mathcal{O})} f(\odot(x))\mathrm{d}\mu(x) = q^{-n} \sum_{x\in X(k)} f(x).$$

## 3. Gauge Measure

In general, $\omega_{X/\mathcal{O}}$ may not have a global section. What makes Weil measure possible is the fact that an $\mathcal{O}$-valued invertible function has constant absolute value 1. If we first base change $X$ to $F$ and try to do an analogous construction, it will usually fail. Of course, if $\omega_{X_F/F}$ happens to be trivial, one has a well-defined, nowhere vanishing measure induced by a global generator of $\omega_{X_F/F}$. In literature such a generator is called a *gauge form*, whose induced measure we may call a *gauge measure*. One can scale the measure by a invertible global function, thus unlike Weil measure, there is no obvious choice for a canonical one.

**Example 3.1.** Let $X = \mathbb{P}^1$, and $[x, y]$ the standard coordinate on $X$, then $X(\mathcal{O}) \simeq X(F)$ naturally. The Weil measure for $X(\mathcal{O})$ is induced by $\mathcal{O}$-differential forms $\mathrm{d}x$ and $\mathrm{d}y$. If we identify $X(\mathcal{O})$ with $F \cup \{\infty\}$, then $\mathrm{d}x$ induces the measure on $\mathcal{O}$ and $\mathrm{d}y$ induces the measure on $(\mathcal{O} \setminus \{0\})^{-1} \cup \{\infty\}$, and they glue on the overlap $\mathcal{O}^\times$. But if we first base change to $F$ and then try to do the similar by regarding $\mathrm{d}x$ and $\mathrm{d}y$ as $F$-differential forms, we have that $\mathrm{d}x$ induces measure on $F$ and $\mathrm{d}y$ on $F^\times \cup \{\infty\}$, but they do not glue on the overlap $F^\times$.

## 4. Link Measures

Both gauge measure and Weil measure can be viewed as special cases of the following definition.

**Definition 4.1.** Let $X$ be a smooth $F$-variety. A real-valued measure $\mu$ on $X(F)$ is called a *Link measure* if one can find a Zariski open cover $\{U_i\}$ of $X$, $s_i \in \omega_{U_i/F}(U_i)$, and $\mu_{s_i}$-measurable sets $A_i \subset U_i(F)$, such that

(1) the cover $\{U_i(F)\}$ of $X(F)$ is locally finite,
(2) $\mu|_{X(F)\setminus(\cup_i A_i)} = 0$,
(3) $s_i$ is nowhere vanishing on $A_i$,
(4) $\mu|_{A_i} = \mu_{s_i}|_{A_i}$.

In this case we write $\mu = \mathfrak{L}\{(U_i, s_i, A_i)\}$.

*Remark* 4.2. If $\mu = \mathfrak{L}\{(U_i, s_i, A_i)\}$ and $\{V_j\}$ is a locally finite (in the open covering sense) Zariski-refinement of $\{U_i\}$,then $\mu = \mathfrak{L}\{(V_j, s_i|_{V_j}, A_i \cap V_j(F))\}$.The local finiteness assumption on the cover may be weakened, but we choose not to do so to avoid unnecessary analysis.

If $X$ is a smooth $\mathcal{O}$-variety, then its Weil measure is a Link measure where $A_i = U_i(\mathcal{O}) \subset U_i(F)$ for suitable Zariski charts $\{U_i\}$. A gauge measure will be the case where there is a single chart $X$ and $A = X(F)$. When the algebro-geometric background gets complicated, it might be confusing to consider $X(\mathcal{O})$ as $\mathcal{O}$-points of $X$, because a point in the sheaf-theoretic sense is not a point in the topological sense. Through Link measure, one is able to view everything including Weil measure in a more topologial setting, thus less prone to mistakes.

## 5. Non-smooth Case

In the case where $X/\mathcal{O}$ is not smooth, one still has numerous choices of Link measures on $X(\mathcal{O})\cap X_F^{\diamondsuit}(F)$, where $X_F^{\diamondsuit}$ denotes the $F$-smooth locus of $X_F$. But in general a random Link measure will not be useful in practice, such as understanding the singularities of $X$. Since this is not the focal point of this paper, we don't expand this further. Readers can refer to say [Yas14] for a particular construction that is useful.

## 6. Fubini Theorem

For a measures $\mu$ on a $p$-adic manifold, let $\operatorname{supp}\mu$ be the support of the measure. There is a relative Fubini theorem for $p$-adic integrals.

**Theorem 6.1.** *Let $\pi\colon X \to Y$ be a smooth map of smooth $F$-varieties, and $\mu_X$ and $\mu_Y$ Link measures on $X(F)$ and $Y(F)$ respectively such that $\pi(\operatorname{supp}\mu_X) \subset \operatorname{supp}\mu_Y$, where* supp *denotes the support. Let $f\colon X(F) \to \mathbb{C}$ be an integrable function. Then for each $y \in Y(F)$, there exists a Link measure $\mu_{X_y}$ on $X_y = \pi^{-1}(y)$, such that*

$$\int_{X(F)} f(x)\mathrm{d}\mu_X(x) = \int_{Y(F)} \left(\int_{X_y(F)} f(x)\mathrm{d}\mu_{X_y}(x)\right) \mathrm{d}\mu_Y(y).$$

*Moreover, Zariski-locally on $X$, the differential form that induces $\mu_{X_y}$ is the restriction of a form in $\omega_{X/Y}$.*

*Proof.* One has the short exact sequence of differential sheaves since $\pi$ is smooth:

$$0 \to \pi^*\Omega_{Y/F} \to \Omega_{X/F} \to \Omega_{X/Y} \to 0,$$

which induces the isomorphism of invertible sheaves $\omega_{X/F} \cong \pi^*\omega_{Y/F} \otimes_{\mathcal{O}_X} \omega_{X/Y}$.

By standard measure theory one can assume $f$ is the characteristic function of a measurable set $S \subset X(F)$. By Remark 4.2, one may assume $\mu_X = \mathbb{1}_A\mu_{\theta_X}$ and $\mu_Y = \mathbb{1}_B\mu_{\theta_Y}$, where $A \subset X(F)$, and $\theta_X \in \omega_{X/F}$ is a nonvanishing global form, and similarly $B \subset Y(F)$ and $\theta_Y \in \omega_{Y/F}$ is a nonvanishing global form too. By assumption we have $\pi(A) \subset B$. Thus by the usual Fubini theorem of measure spaces, for each $y \in Y(F)$ we have $\mu_{X_y} = \mathbb{1}_{S\cap A\cap X_y(F)}\mu_{\theta_X/\theta_Y}$. ▲

*Remark* 6.2. The proof also tells us if $\mu_X$ and $\mu_Y$ are both gauge measures, then so is $\mu_{X_y}$. Similarly, if $\pi$ is induced by $\mathcal{O}$-smooth map between smooth $\mathcal{O}$-varieties, and both $\mu_X$ and $\mu_Y$ are Weil measures, then $\mu_{X_y}$ is also the Weil measure if $y \in Y(\mathcal{O})$, and zero otherwise.

## 7. Flexibility in Point Counting and Caveat

As seen in Theorem 2.3, $p$-adic integral can be used to count $k$-points of smooth $\mathcal{O}$-varieties. One advantage of this method is flexibility, as demonstrated by Proposition 7.2. It uses the following standard result [Igu07].

**Theorem 7.1.** *Let $X$ be an irreducible smooth $F$-variety, $Y$ a subvariety of lower dimension, and $\mu_X$ a Link measure. Then $\mu_X(Y(F)) = 0$. In other words, for any compact open subset $K \subset X(F)$, $\mu_X(K \setminus Y(F)) = \mu_X(K)$.*

Suppose $\pi\colon X \to Y$ is a generically smooth map of irreducible smooth $\mathcal{O}$-varieties. Let $U \subset Y$ be an open dense $\mathcal{O}$-subscheme over which $\pi$ is smooth, and $V = Y \setminus U$ with the reduced scheme structure. Let $y_0 \in V(k)$. One may try to count the number of $X_{y_0}(k)$ by exploiting Theorem 7.1.

**Proposition 7.2.** *One has*

$$\#X_{y_0}(k) = q^{\dim X} \int_{\mathcal{D}_Y(y_0)\setminus V(F)} \int_{X_y(F)} \mathrm{d}\mu_{X_y}\mathrm{d}\mu_Y(y), \tag{7.1}$$

*where $\mu_X$, $\mu_Y$ are Weil measures on $X$ and $Y$ respectively, and $\mu_{X_y}$ is some suitable Link measure.*

*Proof.* Denote $Z = X_{y_0}(k)$, $\widetilde{U} = \pi^{-1}(U)$ and $\widetilde{V} = \pi^{-1}(V)$. Since $X$ is $\mathcal{O}$-smooth and $Z \subset X(k)$, by Theorem 2.3 and Theorem 7.1, we have

$$\#X_{y_0}(k) = q^{\dim X} \int_{\mathcal{D}_X(Z)} \mathrm{d}\mu_X = q^{\dim X} \int_{\mathcal{D}_X(Z)\setminus\widetilde{V}(F)} \mathrm{d}\mu_X.$$

We have $\mathcal{D}_X(Z) \setminus \widetilde{V}(F) \subset \widetilde{U}(F)$, and $\mathcal{D}_Y(y_0) \setminus V(F) \subset U(F)$. One easily sees that $\mu_X$ restricted to $\widetilde{U}_F$ is a Link measure (albeit no longer the Weil measure on $\widetilde{U}_F$), and so is true for $\mu_Y$ restricted to $U(F)$. Since $\widetilde{U}_F \to U_F$ is smooth, by Theorem 6.1 one gets the result. ▲

Proposition 7.2 is generally hard to use because $\mu_{X_y}$ is complicated. A deeper reason behind is that the Weil measure is very lossy in terms of communication between different Zariski charts, as seen in Example 3.1 already. The following is a simple example of what $\mu_{X_y}$ may look like.

**Example 7.3.** Let $A = k[x,y]$. Let $Y = \operatorname{Spec} A$, and $X = \operatorname{Bl}_{(0,0)} Y \cong \operatorname{Proj}_A A[u,v]/(uy - vx)$ be the blow-up of $Y$ at the origin. Denote by $\pi\colon X \to Y$ the natural map. This is a generically smooth map which is an isomorphism over $U \coloneqq Y \setminus \{(0,0)\}$. Cover $X$ with two affine charts $X_1 \coloneqq \operatorname{Spec} k[x,y,\frac{u}{v}]/(y\frac{u}{v} - x)$ and $X_2 \coloneqq \operatorname{Spec} k[x,y,\frac{v}{u}]/(x\frac{v}{u} - y)$. Let $\pi_1 = \pi|_{X_1}$ and $\pi_2 = \pi|_{X_2}$. Let $U_1 = \pi_1^{-1}(U)$ and $U_2 = \pi_2^{-1}(U)$.

The Weil measure on $Y(\mathcal{O})$ is induced by the form $\theta_Y = \mathrm{d}x \wedge \mathrm{d}y$, and that on $X(\mathcal{O})$ is induced by forms $\theta_1 = \mathrm{d}(\frac{u}{v}) \wedge \mathrm{d}y$ on $X_1$ and $\theta_2 = \mathrm{d}x \wedge \mathrm{d}(\frac{v}{u})$ on $X_2$. We have

$$\theta_1|_{U_1} = \frac{1}{y}\pi_1|_{U_1}^*(\theta_Y|_U), \text{ and } \theta_2|_{U_2} = \frac{1}{x}\pi_2|_{U_2}^*(\theta_Y|_U).$$

The coordinate of a point $p \in X_1(\mathcal{O})$ can be denoted by triplet $(x,y,\frac{u}{v})$ where $x,y,\frac{u}{v} \in \mathcal{O}$, and $y\frac{u}{v} = x$. If $(x,y) \neq (0,0)$, then $p \in U_1(F) \cap X_1(\mathcal{O})$. Note in this case $\operatorname{val}_F(x) \geq \operatorname{val}_F(y)$. We have a similar statement for a point in $X_2(\mathcal{O})$. We also have that $p \in X_1(\mathcal{O}) \cap X_2(\mathcal{O})$ if $\frac{u}{v} \in \mathcal{O}^\times$. Combining the information, if $(x,y) \in U(F) \cap Y(\mathcal{O})$, we have the relative measure on the fiber (which is a function on a point, i.e. a number)

$$\mu_{X_{(x,y)}} = \frac{1}{\max\{|x|_F, |y|_F\}}.$$

Thus we see the relative measure on the fiber is complicated if $(x,y)$ specializes to $(0,0)$ even though the map between the underlying manifolds is an (algebraic) isomorphism.

Let $V = \{(0,0)\}$, and $y_0 = (0,0) \in V(k)$. Let $\mu_{F\times F}$ be the additive Haar measure on $F \times F$, normalized so that $\mathcal{O} \times \mathcal{O}$ has volume 1. Then (7.1) gives that

$$\begin{aligned} q^{-2}\#X_{y_0}(k) &= \int_{(x,y)\in\mathcal{D}_Y(y_0)\setminus V(F)} \frac{1}{\max\{|x|_F, |y|_F\}} \mathrm{d}\mu_Y(x,y) \\ &= \sum_{i,j\geq 1} \frac{\mu_{F\times F}(\varpi^i\mathcal{O}^\times \times \varpi^j\mathcal{O}^\times)}{q^{-\min\{i,j\}}} + \sum_{i\geq 1} \frac{\mu_{F\times F}(\{0\}\times\varpi^i\mathcal{O}^\times \cup \varpi^i\mathcal{O}^\times \times \{0\})}{q^{-i}} \\ &= \sum_{i\geq 1} q^i \mu_{F\times F}(\varpi^i\mathcal{O}^\times \times \varpi^i\mathcal{O}^\times) + 2\sum_{i\geq 1} q^i \mu_{F\times F}(\varpi^i\mathcal{O}^\times \times \varpi^{i+1}\mathcal{O}) \\ &= \sum_{i\geq 1} q^i \cdot ((q-1)q^{-i-1})^2 + 2\sum_{i\geq 1} q^i \cdot (q-1)q^{-i-1} \cdot q^{-i-1} \\ &= q^{-2}(q+1). \end{aligned}$$

This in turn shows that $X_{y_0}$ has $q+1$ $k$-points, which we know is true because $X_{y_0} \cong \mathbb{P}^1$.

## 8. A Special Case

In order to utilize Proposition 7.2, one needs additional input.

**Lemma 8.1.** *Let $\pi\colon X \dashrightarrow Y$ be a rational map between irreducible smooth Calabi-Yau $\mathcal{O}$-varieties. Suppose there are open dense $\mathcal{O}$-subschemes $U \subset Y$ and $\widetilde{U} \subset X$ such that $\pi$ is defined and étale on $\widetilde{U}$ with image contained in $U$, and $\operatorname{codim}_X X\setminus\widetilde{U} \geq 2$. Let $\mu_X$ and $\mu_Y$ be the Weil measures on $X(F)$ and $Y(F)$ respectively. Let $y \in Y(\mathcal{O}) \cap U(F)$, and $\mu_{\widetilde{U}_y}$ be the relative measure induced by Theorem 6.1 applied to $(\widetilde{U}_F, \mu_X|_{\widetilde{U}(F)}) \to (U_F, \mu_Y|_{U(F)})$. Then $\mu_{\widetilde{U}_y}$ is the counting measure on $\widetilde{U}_y(F) \cap X(\mathcal{O})$.*

*Proof.* We may assume both $\mu_X$ and $\mu_Y$ are induced by gauge forms $\theta_X$ and $\theta_Y$ (defined over $\mathcal{O}$). Thus $(\mu_X)|_{\widetilde{U}(F)} = \mathbb{1}_{\widetilde{U}(F)\cap X(\mathcal{O})}\mu_{(\theta_X|_{\widetilde{U}})}$ and $(\mu_Y)|_{U(F)} = \mathbb{1}_{U(F)\cap Y(\mathcal{O})}\mu_{(\theta_Y|_U)}$. Since $\pi|_{\widetilde{U}}$ is étale, we have

$$\theta|_{\widetilde{U}} = f\pi|_{\widetilde{U}}^*(\theta_Y|_U)$$

for some invertible function $f$ on $\widetilde{U}$. Since $\operatorname{codim}_X X \setminus \widetilde{U} \geq 2$, one can uniquely extend $f$ to an invertible function on $X$, still denoted by $f$. Let $x \in \widetilde{U}_y(F) \cap X(\mathcal{O})$ then $|f(x)|_F = 1$, which is exactly the value of $\mu_{\widetilde{U}_y}$ on $x$. ▲

**Corollary 8.2.** *Suppose we have the following commutative diagram of smooth irreducible $\mathcal{O}$-varieties:*

$$\begin{array}{ccccc} \widetilde{U} & & \xrightarrow{\pi} & & U \\ \big\downarrow{\scriptstyle i} & & & & \big\downarrow{\scriptstyle j} \\ X & & & & Y \\ & \searrow^{f} & & \swarrow_{g} & \\ & & A & & \end{array},$$

*where $X$ and $Y$ are Calabi-Yau (relative to $\mathcal{O}$), $i$ and $j$ are open embeddings, $f$ is proper, and $\pi$ is étale. Moreover, suppose* $\operatorname{codim}_X X \setminus \widetilde{U} \geq 2$. *Let $\mu_X$ and $\mu_Y$ be the Weil measures on $X(F)$ and $Y(F)$ respectively. Then for any $y \in U(F) \cap Y(\mathcal{O})$, the relative measure on $\widetilde{U}_y(F)$ induced by Theorem 6.1 (applied to $\pi\colon (\widetilde{U}_F, \mu_X|_{\widetilde{U}(F)}) \to (U_F, \mu_Y|_{U(F)})$) is the counting measure.*

*Proof.* Using lemma 8.1, we only need to prove $\widetilde{U}_y(F) \subset X(\mathcal{O})$, but this is easy: let $x \in \widetilde{U}_y(F)$, then $f(x) = g(y) \in A(\mathcal{O})$. Since $f$ is proper, $x \in X(\mathcal{O})$. So we are done. ▲

*Remark* 8.3. It is crucial that $\pi$ is a rational map for lemma 8.1 to be useful, because a result by Gabber [Sta19, Tag 0EA4] shows if $\pi$ is a morphism, it is in fact étale on all of $X$.

## References


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