DEFINITION OF TRANSFER FACTORS IN STANDARD ENDOSCOPY: A SUMMARY

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Contents

1. Introduction

In this short notes, I try to summarize the definition of transfer factors in standard endoscopy thoery in a way more clear for myself to read. The materials are mostly extracted from Langlands and Shelstad [LS87], and some are from Kottwitz [Kot84]. There are some other results, well-known or folklore, that I didn't include a citation since they are more or less standard now (and I'm lazy).

I do try to write the definitions of a-data and χ-data in a more "symmetric" way and to emphasize the duality between them. For some technical parts, I also try to give an even more brief conceptual outline (see § 4.5 for example).

For anyone who is semi-newcomer to this subject like me, a clear and precise understanding of the functorial formulation of L-groups is necessary. This notes gives an account on this matter, assuming familiarity with the absolute theory regarding the *category* of reductive groups.

2. Review on L-groups

2.1. Let F be a local or global field, and G a connected reductive group over F. For convenience, let $\bar{F}$ be the *separable* closure of F, and $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$, and suppose G is split over $\bar{F}$. If F is local, let $|\cdot|_F$ be a fixed absolute value of F. For example, we may choose the "usual" absolute value if $F = \mathbb{R}$ or $\mathbb{C}$, and $|\pi|_F = q^{-1}$ for any uniformizer π of F and q is the order of the residue field of F.

For any two Borel pairs (T_1, B_1) and (T_2, B_2) of G over $\bar{F}$, any inner automorphism of G carrying (T_1, B_1) to (T_2, B_2) induces the same isomorphism $T_1 \to T_2$. So we have canonical isomorphisms on the associated based root datum

$$(\mathbb{X}(T_1), \Delta(T_1, B_1), \check{\mathbb{X}}(T_1), \check{\Delta}(T_1, B_1)) \longrightarrow (\mathbb{X}(T_2), \Delta(T_2, B_2), \check{\mathbb{X}}(T_2), \check{\Delta}(T_2, B_2)).$$

Thus we obtain a diagram of based root data indexed by Borel pairs of G, on which Γ_F acts as automorphisms. One can thus form the limit of this diagram, represented by an abstract based root datum $(\mathbb{X}, \Delta, \check{\mathbb{X}}, \check{\Delta})$, on which Γ_F-acts. We shall call this the *canonical based root datum* of

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G, denoted by $\Psi_0(G)$. The Weyl group together with its simple reflections attached to $\Psi_0(G)$ is denoted by $\Omega_0(G)$. Note that both $\Psi_0(G)$ and $\Omega_0(G)$ encodes the Γ_F-action.

More generally, if Γ is any group acting on G over F, then Γ acts on $\Psi_0(G)$ and $\Omega_0(G)$as well.

2.2. Take the Γ_F-dual of $\Psi_0(G)$, because $\mathbb{C}$ is algebraically closed, we can obtain a $\mathbb{C}$-reductive group $\check{G}$ with a splitting $(\check{\mathbf{T}}, \check{\mathbf{B}}, \{X_{\check{\alpha}}\})$, on which Γ_F acts. The Weyl group $\Omega_{\check{\mathbf{T}}}$ of $\check{\mathbf{T}}$ in $\check{G}$ may be identified with $\Omega_0(G)$, compatible with Γ_F-action.

The Weil group W_F acts on these objects via projection $W_F \to \Gamma_F$, and we can form L-group

$$^LG = \check{G} \rtimes W_F.$$

2.3. If G is quasi-split over F, fix an F-splitting $(\mathbf{T}, \mathbf{B}, \{X_\alpha\})$ of G. The resulting based root system is $(\mathbb{X}(\mathbf{T}), \Delta(\mathbf{T}, \mathbf{B}), \check{\mathbb{X}}(\mathbf{T}), \check{\Delta}(\mathbf{T}, \mathbf{B}))$ on which Γ_F acts, and in this case it may be identified with $\Psi_0(G)$. Let $\Omega_{\mathbf{T}}$ be the Weyl group of $(G, \mathbf{T})$, then Γ_F acts on $\Omega_{\mathbf{T}}$ as well. We may also identify $\Omega_0(G)$ with $\Omega_{\mathbf{T}}$ that is compatible with Γ_F-action.

2.4. If a group Γ acts on a reductive group H over an algebraically closed field, say $\mathbb{C}$, the resulting Γ-action on $\Psi_0(H)$ induces a Γ-action on some splitting of H, hence also an(other) action of Γ on H. However, the two actions may not coincide, and the original Γ-action may not fix a splitting of H at all. Therefore we have the following definition.

Definition 2.1. Let H be a reductive group over $\mathbb{C}$, and Γ a group acting on H. Such action is called an *L-action* if it stabilizes a splitting.

The action of W_F on $\check{G}$ in forming ^LG is thus an L-action by definition.

Suppose we have a split extension

$$1 \longrightarrow \check{G} \longrightarrow \mathscr{G} \longrightarrow W_F \longrightarrow 1,$$

then $\mathscr{G}$ is not necessarily an L-group since W_F-action induced by a splitting of this extension may not be an L-action. Nonetheless, for a fixed Γ_F-splitting of $\check{G}$, we may attach to this extension an L-action. Let $c\colon W_F \to \mathscr{G}$ be any splitting of this extension, then it induces map $W_F \to \mathrm{Aut}(\check{G})$, hence $W_F \to \mathrm{Out}(\check{G})$, the latter depending only on the extension but not c. Using a fixed Γ_F-splitting **Spl**, we may identify $\mathrm{Out}(\check{G})$ with the subgroup of $\mathrm{Aut}(\check{G})$ that fixes **Spl**. Therefore we obtain an L-action of W_F on $\check{G}$. We will call this the L-action associated with $\mathscr{G}$ and splitting **Spl**. We shall use the earlier fixed splitting $(\check{\mathbf{T}}, \check{\mathbf{B}}, \{X_{\check{\alpha}}\})$ for **Spl**.

3. Abstract Cohomological Formulations

3.1. Let X be a $\mathbb{Z}$-lattice, and $R \subset X$ a finite subset such that $-R = R$. A *gauge* p of R is a map $R \to \{\pm 1\}$ such that $p(-\alpha) = -p(\alpha)$ for all $\alpha \in R$.

For example, if $R \subset X = \mathbb{X}(T)$ be the root system for maximal torus $T \subset G$, and B a Borel containing T, then B determines a gauge p_B on R such that $p_B(\alpha) = 1$ if and only if α is a root of T in B.

If $O \subset R$ be a subset such that $R = O \coprod -O$ is a disjoint union, then one can define gauge $p_O(\alpha) = 1$ if and only if $\alpha \in O$.

3.2. Let $\Sigma = \Gamma \times \langle \epsilon \rangle$ be a group acting on X and R such that ϵ acts as -1. In our applications we will have $\epsilon^2 = 1$ so we will assume this as well, even though it's not required in many of the results below. Then we define a product notation for any r-tuple $a = (a_1, \dots, a_r) \in \Sigma^r$

$$\prod_{\alpha : a}^{p} = \prod_{\alpha : a_1, \dots, a_r}^{p} := \prod_{\substack{\alpha \in R \\ p((a_1 \cdots a_s)^{-1}\alpha) = (-1)^{s+1} \\ 1 \le s \le r}}$$

Let k be a field and let Σ acts on k trivially, then it acts on $k^\times \otimes_{\mathbb{Z}} X$, and we denote $c \otimes \lambda$ by c^λ.

Lemma 3.1. *The* 2*-cochain*

$$t_p(\sigma,\tau) = \prod_{\alpha:1,\sigma,\tau}^{p} (-1)^\alpha$$

is a 2*-cocycle* $\Sigma^2 \to k^\times \otimes_{\mathbb{Z}} X$. *Moreover, if* q *is another gauge,* t_p/t_q *is a coboundary.*

3.3. If p is a gauge, then so is $-p$, and we define for a pair of gauge (p,q) and an r-tuple $a \in \Sigma^r$ another product

$$\prod_{\alpha:a}^{p,q} = \prod_{\alpha:a_1,\dots,a_r}^{p,q} \coloneqq \prod_{\substack{\alpha\in R\\ p((a_1\cdots a_s)^{-1}\alpha)=(-1)^{s+1}\\ q((a_1\cdots a_s)^{-1}\alpha)=1\\ 1\le s\le r}}$$

Then if we define 1-cochain of Γ

$$s_{p/q}(\sigma) = \prod_{\alpha:1,\sigma}^{p,q} (-1)^\alpha \prod_{\alpha:1,\sigma}^{-q,p} (-1)^\alpha,$$

then one can show that

$$\partial s_{p/q} = t_p/t_q,$$

as cochains of Γ (not Σ).

4. Story on G-side: a-data

4.1. Let G be quasi-split and fix a splitting as before. Let U_α be the root groups of G such that $X_\alpha = U_\alpha(1)$. We can define

$$\begin{aligned} n\colon \Omega_{\mathbf{T}} \rtimes \Gamma_F &\longrightarrow \mathrm{N}_G(\mathbf{T}) \rtimes \Gamma_F \\ w \rtimes \sigma &\longmapsto n(w) \rtimes \sigma, \end{aligned}$$

where $n(w)$ is such that if $w = s_{\alpha_1}\cdots s_{\alpha_r}$ is a reduced expression, then

$$\begin{aligned} n(w) &= n(s_{\alpha_1})\cdots n(s_{\alpha_r}),\\ n(s_\alpha) &\coloneqq U_\alpha(1)U_{-\alpha}(-1)U_\alpha(1),\\ n(1) &\coloneqq 1. \end{aligned}$$

One can show that $n(w)$ is independent of the reduced expression hence is well defined, and that $n\colon \Omega_{\mathbf{T}} \to \mathrm{N}_G(\mathbf{T})$ is Γ_F-equivariant. Therefore for $\theta \in \Omega_{\mathbf{T}} \rtimes \Gamma_F$, $n(\theta)$ acts on $\mathbf{T}$ as θ, and

$$t(\theta_1,\theta_2) \coloneqq n(\theta_1)n(\theta_2)n(\theta_1\theta_2)^{-1}$$

is a 2-cocycle of $\Omega_{\mathbf{T}} \rtimes \Gamma_F$ in $\mathbf{T}(\bar{F})$.

Lemma 4.1. *We have that*

$$t(\theta_1,\theta_2) = t_{p_B}(\theta_1,\theta_2) = \prod_{\check{\alpha}:1,\theta_1,\theta_2}^{p_{\mathbf{B}}} (-1)^{\check{\alpha}},$$

where $p_{\mathbf{B}}$ *is the gauge determined by* $\mathbf{B}$ *on* $\check{R}(G,\mathbf{T})$.

Note that the definition of t depends on root vectors X_α, while the right-hand side of the Lemma above doesn't.

4.2. The 2-cocycle t is in fact a coboundary when restricted to certain subgroup of $\Omega_{\mathbf{T}} \rtimes \Gamma_F$, and a splitting can be found with the help of so called a-data. The abstract formulation is as follows. Retain notations in the subsection about Σ acting on X and R. Suppose Σ acts on $\bar{k}/k$ such that ϵ still acts trivially (but not necessarily for Γ).

Definition 4.2. An *a-datum* is a Γ-equivariant map

$$\begin{aligned} a\colon R &\longrightarrow \bar{k}^\times \\ \alpha &\longmapsto a_\alpha \end{aligned}$$

such that $a_{-\alpha} = -a_\alpha$ (i.e. a is "ϵ-antivariant"). A *b-datum* is a Γ-equivariant map $b\colon R \to \bar{k}^\times$ that is also ϵ-equivariant (hence Σ-equivariant).

Suppose a-data exists for Σ-action on R, and p a gauge of R, then we form 1-cochain of Γ

$$u_p(\sigma) \coloneqq \prod_{\alpha:1,\sigma}^{p} a_\alpha^\alpha \in \bar{k}^\times \otimes_{\mathbb{Z}} X.$$

Lemma 4.3. *Viewing t_p as a 2-cocycle of Γ with value in $\bar{k}^\times \otimes_{\mathbb{Z}} X \supset k^\times \otimes_{\mathbb{Z}} X$, we have that*

$$\partial u_p = t_p.$$

Similarly, if b-data exists, we can form 1-cochain that is in fact a cocycle:

$$v_p(\sigma) \coloneqq \prod_{\alpha:1,\sigma}^{p} b_\alpha^\alpha \in \bar{k}^\times \otimes_{\mathbb{Z}} X.$$

4.3. To emphasize the duality to χ-data later, here we use F instead of k. Let $\Gamma = \Gamma_F$. For $\alpha \in R$, let F_α be its splitting field, and $F_{\pm\alpha}$ be the splitting field of $\pm\alpha$. Let Γ_α and $\Gamma_{\pm\alpha}$ be their respective absolute Galois group. Then $[F_\alpha : F_{\pm\alpha}]$ is 1 if Γ_F-orbit of α doesn't contain $-\alpha$ or 2 otherwise.

Suppose we have a-data for Γ_F-action on R. Then we always have $a_\alpha \in F_\alpha^\times$. If moreover $[F_\alpha : F_{\pm\alpha}] = 2$, we must have that $\sigma(a_\alpha) = -a_\alpha$ for the unique non-trivial element $\sigma \in \Gamma_{\pm\alpha}/\Gamma_\alpha$. This means $a_\alpha^2 = -1$ viewed as elements of group $F_\alpha^\times / \mathrm{Nm}_{F_\alpha/F_{\pm\alpha}}(F_\alpha^\times)$.

4.4. Let $T \subset G$ be a maximal F-torus. Let $h \in G(\bar{F})$ be a chosen transporter from $\mathbf{T}$ to T, i.e. $\mathrm{Ad}_h(\mathbf{T}) = T$. Then $h^{-1}\sigma(h)$ acts on $\mathbf{T}$ by conjugation, hence $h^{-1}\sigma(h) \in \mathrm{N}_G(\mathbf{T})$, whose image in $\Omega_{\mathbf{T}}$ is denoted by $\omega_T(\sigma)$. Thus if we denote by σ_T the action of σ on $\mathbf{T}$ by transporting that on T to $\mathbf{T}$ using h, then $\sigma_T = \omega_T(\sigma) \rtimes \sigma \in \Omega_{\mathbf{T}} \rtimes \Gamma_F$. Let Γ_T be the group generated by σ_T. Clearly σ_T depends only on the choice of $B = \mathrm{Ad}_h(\mathbf{B})$, not h itself.

The action of $\Sigma = \Gamma_F \times \langle\epsilon\rangle$ on $\check{R}(G,T) \subset \check{\mathbb{X}}(T)$ admits a-data, which transports to $\Gamma_T \times \langle\epsilon\rangle$-action on $R(G,\mathbf{T})$. Let $\{a_{\check{\alpha}}\}$ be an a-datum. We have gauge $p = p_{\mathbf{B}}$ on $\check{R}(G,\mathbf{T})$, so we have

$$x_p(\sigma_T) = \prod_{\check{\alpha}:1,\sigma_T}^{p} a_{\check{\alpha}}^{\check{\alpha}},$$

whose coboundary is

$$t_p(\sigma_T,\tau_T) = n(\sigma_T)n(\tau_T)n(\sigma_T\tau_T)^{-1}.$$

Since $a_{\check{\alpha}}^{-1}$ is also an a-datum, we also have that

$$\partial x_p^{-1} = t_p.$$

Thus the map

$$\begin{aligned} \Gamma_T &\longrightarrow \mathrm{N}_G(\mathbf{T}) \rtimes \Gamma_F \\ \sigma_T &\longmapsto x_p(\sigma_T)n(\sigma_T) \end{aligned}$$

is a homomorphism, hence induces 1-cocycle $\sigma_T \mapsto x_p(\sigma_T)n(\omega_T(\sigma)) \eqqcolon m(\sigma_T)$.

Transporting by $\mathrm{Ad}_{h\rtimes 1}$ inside $G\rtimes\Gamma_F$, one has map

$$\begin{aligned}\Gamma_F &\longrightarrow \mathrm{N}_G(T)\rtimes\Gamma_F\\ \sigma &\longmapsto hx_p(\sigma_T)n(\sigma_T)h^{-1}=hm(\sigma_T)\sigma(h)^{-1}\rtimes\sigma,\end{aligned}$$

whose image lies in $T\rtimes\Gamma_F$. One thus obtains a 1-cocycle of Γ_F in T, whose cohomology class depends only possibly on $B=\mathrm{Ad}_h(\mathbf{B})$, not h. In fact, a long computation would show it doesn't depend on B either. Call this cohomology class in $\mathrm{H}^1(F,T)$ by λ_T.

4.5. Here we try to summarize the construction of λ_T using a-data more conceptually. To begin with, we have extension

$$1\longrightarrow \mathbf{T}\longrightarrow \mathrm{N}_G(\mathbf{T})\rtimes\Gamma_F\longrightarrow \Omega_{\mathbf{T}}\rtimes\Gamma_F\longrightarrow 1. \tag{4.5.1}$$

The choice of a Γ_F-equivariant set-theoretic section $n\colon\Omega_{\mathbf{T}}\to\mathrm{N}_G(\mathbf{T})$ gives a set-theoretic section $n\times\mathrm{id}$ of this extension, which in turn gives a 2-cocycle of $\Omega_{\mathbf{T}}\rtimes\Gamma_F$ in $\mathbf{T}$. We still use n instead of $n\times\mathrm{id}$ for convenience.

Given T, we choose B containing T, and $h\in G$ such that $\mathrm{Int}\,h$ maps $(\mathbf{T},\mathbf{B})$ to (T,B). Then we obtain another splitting of $\Omega_{\mathbf{T}}\rtimes\Gamma_F$ via map $\Gamma_F\to\Gamma_T$. Restricting the extension (4.5.1) to Γ_T, we have

$$\begin{array}{ccccccccc}1&\longrightarrow&\mathbf{T}&\longrightarrow&\mathbf{N}_T&\longrightarrow&\Gamma_T&\longrightarrow&1\\ &&\|&&\downarrow&&\downarrow&&\\ 1&\longrightarrow&\mathbf{T}&\longrightarrow&\mathrm{N}_G(\mathbf{T})\rtimes\Gamma_F&\longrightarrow&\Omega_{\mathbf{T}}\rtimes\Gamma_F&\longrightarrow&1\end{array},$$

where the square on the right is Cartesian. This extension of Γ_T is split, and a choice of an a-datum provides a splitting x_pn, whose composition with (set-theoretic) projection $\mathbf{N}_T\to\mathrm{N}_G(\mathbf{T})$ gives 1-cocycle m.

Transporting using $\mathrm{Int}\,h$, one has split extension

$$1\longrightarrow T\longrightarrow N_T\coloneqq h\mathbf{N}_Th^{-1}\longrightarrow\Gamma_F\longrightarrow 1\ ,$$

where in fact $N_T=T\rtimes\Gamma_F\subset\mathrm{N}_G(T)\rtimes\Gamma_F$. The difference between the natural splitting $T\rtimes\Gamma_F$ and the splitting obtained using an a-datum gives a class $\lambda_T\in\mathrm{H}^1(F,T)$. The effect of all the choices in the construction of λ_T as well as its various naturalities can be summarized as follows:

(1) it doesn't depend on h or B,
(2) a change in splitting $(\mathbf{T},\mathbf{B},\{X_\alpha\})$ modifies λ_T by an element in the image of map
$$\mathrm{coker}[G(F)\to G_{\mathrm{ad}}(F)]\to\mathrm{H}^1(F,Z)\to\mathrm{H}^1(F,T),$$
(3) a change in a-data resulting a b-datum by taking quotient of two a-data. Forming 1-cocycle v_p using the same definition as x_p but with a-data replaced with b-data (so it is indeed a cocycle), then λ_T is modified by hv_ph^{-1}.
(4) The construction of λ_T is compatible with conjugation of triples $(T,B,\{a_\alpha\})$.
(5) Finally, if instead F is global, one can carry out the same construction for F, and for any place v of F, $\lambda_{v,T}$ is precisely the image of λ_T.

My guess: the class λ_T is the obstruction of lifting T to an F-splitting.

5. Story on $\check{G}$-side: χ-data

5.1. On the dual side we have ${}^LG$ instead of $G\rtimes\Gamma_F$ and everything is dualized. In particular, we have the dual notion to a-data called χ-data. We don't need to assume G to be quasi-split, but we fix a splitting of $\check{G}$ as before.

The abstract formulation of χ-data is as follows. Recall we have $\Sigma=\Gamma\times\langle\epsilon\rangle$-action on X and R. Unlike a-data, since arithmetic duality will be used, χ-data can only be formulated for $\Gamma=\Gamma_F$ where F a local or global field, as we shall assume so. Suppose the action of Γ on X is continuous with respect to the profinite topology on Γ and discrete topology on X. We use C_F to denote either the multiplicative group if F is local or the idele class group if F is global. Let $\hat{\bullet}$ denote Pontryagin dual.

Then for $\alpha \in R$, we have splitting fields $F_\alpha = F_{-\alpha}$ and $F_{\pm\alpha}$, Galois groups Γ_α, $\Gamma_{\pm\alpha}$, as well as groups C_α, $C_{\pm\alpha}$, etc. Note by continuity F_α is finite over F. Since F is local or global, we also have Weil groups W_F, $W_\alpha = W_{F_\alpha}$, and $W_{\pm\alpha} = W_{F_{\pm\alpha}}$. Then Γ acts on $\coprod_{\alpha\in O} \widehat{C_\alpha}$ for any Γ-stable subset $O \subset R$ by $\sigma(\chi_\alpha) = \chi_\alpha \circ \sigma^{-1}$ for any $\sigma \in \Gamma$.

Definition 5.1. A χ-datum is defined to be a Γ-equivariant map

$$\chi\colon R \longrightarrow \coprod_{\alpha\in R} \widehat{C_\alpha},$$

such that $\chi_{-\alpha} = \chi_\alpha^{-1}$ and χ_α is non-trivial on $C_{\pm\alpha}$ if $[F_\alpha : F_{\pm\alpha}] = 2$. A ζ-datum is a map of the same definition as a χ-datum except that ζ_α is trivial on $C_{\pm\alpha}$ if $[F_\alpha : F_{\pm\alpha}] = 2$.

Note that if $[F_\alpha : F_{\pm\alpha}] = 2$, and let $\sigma \in \Gamma_{\pm\alpha}$ be a non-trivial representative of $\Gamma_{\pm\alpha}/\Gamma_\alpha$, then $\sigma(\alpha) = \sigma^{-1}(\alpha) = -\alpha$, and $\chi_\alpha^{-1} = \chi_{-\alpha} = \chi_{\sigma^{-1}(\alpha)} = \chi_\alpha \circ \sigma$. Thus χ_α must be trivial on $\mathrm{Nm}_{C_\alpha/C_{\pm\alpha}}(C_\alpha)$, hence must be a extension of the quadratic quasi-character of $C_{\pm\alpha}$ associated with $F_\alpha/F_{\pm\alpha}$. In addition, we may regard χ_α as a character of W_α via Artin reciprocity.

5.2. Recall that for any gauge p on R we have a 2-cocycle t_p of Σ with value in $k^\times \otimes_{\mathbb{Z}} X$ where $k^\times$ is any field. Let $k = \mathbb{C}$, then we obtain a 2-cocycle of Γ in $\mathbb{C}^\times \otimes_{\mathbb{Z}} X$. This cocycle is in general cohomologically non-trivial, but becomes cohomologically trivial if inflated to W_F. The splitting is given by any χ-datum. Suppose O is a Σ-orbit in R, and $\alpha \in O$ a fixed element. Since $[F_\alpha : F]$ is finite, we can choose a finite set of representatives of $W_{\pm\alpha}\backslash W_F$, denoted by $w_1, \dots, w_n$, whose images $\sigma_1, \dots, \sigma_n$ is a set of representatives of $\Gamma_{\pm\alpha}\backslash\Gamma$. Then $O = \{\pm\sigma_i^{-1}\alpha \mid 1 \le i \le n\}$. Define gauge p on O by declaring $p(\sigma_i^{-1}\alpha) = 1$. We can then assemble p for all orbits O in R to obtain a gauge p on R.

Still fix α and O, we define contraction maps $u_i\colon W \to W_{\pm\alpha}$ for $1 \le i \le n$ by letting $u_i(w) = W_{\pm\alpha}$ to be the element such that

$$w_i w = u_i(w) w_j$$

for appropriate $1 \le j \le n$.

Choose representatives $v_0 \in W_\alpha$ and if $[F_\alpha : F_{\pm\alpha}] = 2$ an element $v_1 \in W_{\pm\alpha} - W_\alpha$. Define contraction $v\colon W_{\pm\alpha} \to W_\alpha$ by

$$v_0 u = v(u) v_j$$

for $j = 0$ or 1 as appropriate. Note if we choose v_0 in the center of W_α, then v is identity when restricted to W_α.

Define 1-cochains of W_F in $\mathbb{C}^\times \otimes_{\mathbb{Z}} X$ by

$$r_{O,p}(w) = \prod_{i=1}^{n} \chi_\alpha(v(u_i(w)))^{\sigma_i^{-1}\alpha},$$

and

$$r_p = \prod_{O\in R/\Sigma} r_{O,p}.$$

For any gauge q on R, we let

$$r_q = s_{q/p} r_p.$$

Lemma 5.2. *We have $\partial r_q = t_q$ as cocycles of W_F for any gauge q. Moreover, for a fixed χ-datum, all choices involved in constructing r_q only change it by a coboundary.*

Similarly, we may replace χ-data with ζ-data and form 1-cocycles (not just cochains) c_p. Since c_p is already a cocycle, we don't need to define c_q hence we use $c = c_p$. Agian for a fixed ζ-datum, all choices made in the construction have no effect on the cohomology class of c.

5.3. Return to the concrete setting of reductive group G. Again T is a maximal F-torus of G. We have the fixed Γ_F-splitting $(\check{\mathbf{T}}, \check{\mathbf{B}}, \{X_{\check{\alpha}}\})$ of $\check{G}$.

An embedding $\xi\colon {}^L T \to {}^L G$ is called *admissible* if

(1) it induces isomorphism $\check{T} \to \check{\mathbf{T}}$ that is the same as the one induced by some choice of Borel B containing T and $\check{\mathbf{B}}$,
(2) it is a morphism of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \check{T} & \longrightarrow & {}^L T & \longrightarrow & W_F & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \check{G} & \longrightarrow & {}^L G & \longrightarrow & W_F & \longrightarrow & 1 \end{array}.$$

Note that $\check{T} \to \check{\mathbf{T}}$ is *not* W_F-equivariant in general, and the $\check{G}$-conjugacy class of ξ is independent of B or $(\mathbf{T}, \mathbf{B})$. We will attach to each χ-datum for the Γ_F-action on $R(G, T)$ an admissible embedding $\xi\colon {}^L T \to {}^L G$, whose $\check{G}$ conjugacy class is canonical.

To start we fix a Borel B containing T. Thus we can transfer the action of Γ_F on $\mathbb{X}(T)$ to an action on $\check{\mathbb{X}}(\check{\mathbf{T}})$, through $\Psi_0(G)$. Thus we obtain an embedding $\Gamma_F \to \Omega_{\check{\mathbf{T}}} \rtimes \Gamma_F$, hence a set-theoretic map $\omega_T\colon \Gamma_F \to \Omega_{\check{\mathbf{T}}}$ and we can inflate it to W_F. Let $W_T \subset \Omega_{\check{\mathbf{T}}} \rtimes W_F$ be the subgroup of elements $\omega_T(w) \rtimes w$ where $w \in W_F$. The construction $n\colon \Omega_{\mathbf{T}} \to \mathrm{N}_G(\mathbf{T})$ also makes sense on the dual side, so we have

$$\check{n}\colon \Omega_{\check{\mathbf{T}}} \longrightarrow \mathrm{N}_{\check{G}}(\check{\mathbf{T}})$$

that is Γ_F-equivariant, hence also W_F-equivariant. Still use $\check{n}$ to denote the map $\check{n} \times \mathrm{id}\colon \Omega_{\check{\mathbf{T}}} \rtimes W_F \to \mathrm{N}_{\check{G}}(\check{\mathbf{T}}) \rtimes W_F$.

Let $p = p_{\check{\mathbf{B}}}$ be the gauge on $\check{R}(\check{G}, \check{\mathbf{T}})$ determined by $\check{\mathbf{B}}$. Then the cocycle of $\Omega_{\check{\mathbf{T}}} \rtimes W_F$, with value in $\check{T}(\mathbb{C})$

$$t_p(w_1, w_2) = \check{n}(w_1)\check{n}(w_2)\check{n}(w_1 w_2)^{-1}$$

is a coboundary when restricted to W_T. A choice of a χ-datum $\{\chi_\alpha\}$ of Γ_F-action (hence W_F-action) on $R(G, T)$ transports to a χ-datum of W_T-action on $\check{R}(\check{G}, \check{\mathbf{T}})$. Note $\{\chi_\alpha^{-1}\}$ is also a χ-datum, and we use it to form 1-cochain r_p^{-1}, so that $\partial r_p^{-1} = t_p$. Thus we obtain homomorphism

$$\begin{aligned} \xi\colon {}^L T &\longrightarrow {}^L G \\ t \rtimes w &\longmapsto t_{B,\check{\mathbf{B}}} r_p(w) \check{n}(w), \end{aligned}$$

where $t \mapsto t_{B,\check{\mathbf{B}}}$ is the map induced by choice of B, and $\check{\mathbf{B}}$.

5.4. To summarize the construction more simply: we are basically embedding the action of W_T on $\check{\mathbf{T}}$, which contains the same information as ${}^L T$, into $\mathrm{N}_{\check{G}}(\check{\mathbf{T}}) \rtimes W_F \subset {}^L G$, an object formed using the "standard" (relative to a choice of splitting anyway) action of W_F.

The effect of various choices and naturalities of ξ can be summarized as follows:

(1) for fixed Γ_F-splitting, choice of B, and choice of χ-data, ξ is determined upto $\check{\mathbf{T}}$-conjugacy,
(2) change of Γ_F-splitting will change ξ by $\mathrm{Int}\, g$ for some $g \in \check{G}^{\Gamma_F}$,
(3) change of B into $B' = vBv^{-1}$ where $v \in \mathrm{N}_G(T)$ will change ξ in the following way: $\mathrm{Int}\, v$ acts on T hence on $\check{T}$, and thus on $\check{\mathbf{T}}$ using ξ. Call this action μ. Let $g \in \mathrm{N}_{\check{G}}(\check{T})$ acts on $\check{T}$ as μ, then ξ' obtained using B' is equal to $\mathrm{Int}\, g^{-1} \circ \xi$.
(4) change of χ-data results in a ζ-datum by taking quotient, then ξ is multiplied by the cocycle c obtained from that ζ-datum, i.e. $\xi'(t \rtimes w) = c(w)\xi(t \rtimes w)$,

(5) if $\operatorname{Int} g$ transports $(T, \{\chi_\alpha\})$ to $(T', \{\chi'_\alpha\})$, then ξ' is simply the composition of ξ with canonical map ${}^L T' \to {}^L T$ induced by $\operatorname{Int} g$,
(6) finally, if F is global, there are two ways to pass from global to local an admissible embedding attached to a global χ-datum: one is directly via map $W_{F_v} \to W_F$ for any place v, and the other is by naturally induce a local χ-datum from the global one, and then attach a local admissible embedding to the χ-datum. One can show there is a choice of those auxiliary data on the way so that these two ways coincide.

6. Endoscopic Groups and Transfer Factors

6.1. Recall we have for an F-group G the L-group ${}^L G$. Let G^* a quasi-split inner form of G, and $\psi\colon G \to G^*$ a fixed inner twist. Then ψ induces isomorphism of L-groups

$$ {}^L\psi\colon {}^L G^* \longrightarrow {}^L G. $$

6.2. An endoscopic datum is a quadruple $(H, \mathscr{H}, s, \xi)$ where

(1) $s \in \check{G}$ is semisimple,
(2) H is quasi-split reductive over F, with L-group ${}^L H$ and a fixed Γ_F-splitting **Spl**,
(3) $\mathscr{H}$ is a split extension of W_F by $\check{H}$, whose associated L-action (for **Spl**) is the same as the one for ${}^L H$,
(4) $\xi\colon \mathscr{H} \to {}^L G$ is an L-embedding, i.e. a morphism of extensions

$$ \begin{array}{ccccccccc} 1 & \longrightarrow & \check{H} & \longrightarrow & \mathscr{H} & \longrightarrow & W_F & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \check{G} & \longrightarrow & {}^L G & \longrightarrow & W_F & \longrightarrow & 1 \end{array}, $$

such that the isomorphic image of $\check{H}$ is equal to $\mathrm{C}_{\check{G}}(s)_0$ (the connected centralizer), and that $\operatorname{Int} s \circ \xi = a\xi$, where a is a cohomologically trivial 1-cocycle of W_F in $Z(\check{G})$, inflated to $\mathscr{H}$.

In this notes $\mathscr{H}$ will be limited to ${}^L H$ for simplicity. Maybe the general case will be added later.

6.3. Given G and endoscopic datum $(H, \mathscr{H}, s, \xi)$, one can construcrt a canonical map from the semisimple conjugacy classes of $H(\bar{F})$ to those of $G(\bar{F})$. Indeed, one chooses a Borel pair (T, B) in G and one $(\mathscr{T}, \mathscr{B})$ in $\check{G}$, which will identify the Weyl group $\Omega(G, T)$ with $\Omega(G, \mathscr{T})$, through $\Psi_0(G)$ and $\Psi_0(\check{G}) = \check{\Psi}_0(G)$ (without concerning Γ_F-action at this point).

Similarly for H we have (T_H, B_H) and $(\mathscr{T}_H, \mathscr{B}_H)$, and identification $\Omega(H, T_H) \simeq_{B_H, \mathscr{B}_H} \Omega(\check{H}, \mathscr{T}_H)$. The embedding $\xi\colon \mathscr{H} \to {}^L G$ gives embedding $\xi\colon \check{H} \to \check{G}$. We can find $x \in \check{G}$ such that $\operatorname{Int} x \circ \xi$ maps $\mathscr{T}_H$ isomorphically onto $\mathscr{T}$. Thus we have isomorphisms (depending on a lot of choices, and doesn't play with Γ_F in general)

$$ \check{T}_H \xrightarrow{\sim} \mathscr{T}_H \xrightarrow{\sim} \mathscr{T} \xrightarrow{\sim} \check{T}, $$

and thus an isomorphism $T_H \xrightarrow{\sim} T$. On the other hand, $\operatorname{Int} x \circ \xi$ also embeds $\Omega(\check{H}, \mathscr{T}_H)$ into $\Omega(\check{G}, \mathscr{T})$, hence $\Omega(H, T_H)$ into $\Omega(G, T)$. Therefore we have a map

$$ T_H/\Omega(H, T_H) \longrightarrow T/\Omega(G, T). $$

By a well-known result of Steinberg, it induces a map

$$ \mathscr{A}_{H/G}\colon \mathrm{Cl}_{\mathrm{ss}}(H(\bar{F})) \longrightarrow \mathrm{Cl}_{\mathrm{ss}}(G(\bar{F})). $$

In the case $G = G^*$ and T_H is defined over F, we may in fact choose (T, B) (without affecting other choices) so that T, and $T_H \to T$ are both defined over F. In this case we call $T_H \to T$ *admissible*.

Since all choices made in constructing $\mathscr{A}_{H/G}$ can only be changed using inner automorphisms, we see $\mathscr{A}_{H/G}$ is canonical. In fact, $\mathscr{A}_{H/G}$ is defined over F, or equivalently Γ_F-equivariant. So if a class $[\gamma_H] \in \mathrm{Cl}_{\mathrm{ss}}(H(\bar{F}))$ is represented by $\gamma_H \in H(F)$, then $\mathscr{A}_{H/G}([\gamma_H]) = [\gamma]$ for some $\gamma \in G(F)$.

Definition 6.1. An element in either $H(F)$ or $G(F)$ is called *strongly regular semisimple* if its centralizer is a torus. An element $\gamma_H \in H(F)$ is called *(resp. strongly) G-regular semisimple* if it is semisimple, and $\mathscr{A}_{H/G}([\gamma_H])$ is a (resp. strongly) regular semisimple class.

A (strongly) G-regular semisimple element is necessarily (strongly) regular semisimple.

6.4. Let F be local. We will now describe the transfer factors for G and endoscopic datum $(H, {}^L H, s, \xi)$. From now on subscript $\bullet_H$ denotes objects related to H, $\bullet_G$ their counterparts related to G, and no subscript means those for G^*. Let G_{sc} be the simply-connected cover of G^*, and subscript $\bullet_{\mathrm{sc}}$ denotes liftings to G_{sc} of objects related to G^*.

Let $\gamma_H \in H(F)$ be strongly G-regular semisimple, and $\gamma_G \in G(F)$ strongly regular semisimple, and $\mathscr{A}_{H/G}([\gamma_H]) = [\gamma_G]$. Let $T_H = \mathrm{C}_H(\gamma_H)$, and $T_H \to T \subset G^*$ an admissible embedding. Let γ be the image of γ_H in T. We fix a a-datum and a χ-datum for Γ_F-action on $R(G^*, T)$ (equivalently $\check{R}(G^*, T)$).

Without loss of generality, we may assume the choice of $(\mathscr{T}, \mathscr{B})$ and $(\mathscr{T}_H, \mathscr{B}_H)$ in constructing embedding $T_H \to T$ is the same as the one that is part of a fixed Γ_F-splitting of $\check{G}$ and $\check{H}$ respectively. In other words, $\mathscr{T} = \check{\mathbf{T}}$, $\mathscr{B} = \check{\mathbf{B}}$ and so on. We may even assume ξ maps $\mathscr{T}_H$ to $\mathscr{T}$ and $\mathscr{B}_H$ into $\mathscr{B}$. In this way we have $s \in \mathscr{T}$, whose image in $\check{T}$ is denoted s_T. Since s is central in $\xi(\check{H})$, s_T depends only on $T_H \to T$ (in particular, independent of B_H) after the explicit choices made on the dual side.

The embedding $Z(\check{G}) \to \check{T}$ is canonical, thus allows us to define $\check{T}_{\mathrm{ad}} = \check{T}/Z(\check{G})$, which is canonically isomorphic to the dual torus of T_{sc}. By definition, the image of s_T in $\check{T}_{\mathrm{ad}}$ is Γ-invariant, hence gives a well-defined element $\mathbf{s}_T \in \pi_0(\check{T}_{\mathrm{ad}}^{\Gamma_F})$.

6.5. Fix once and for all an F-splitting **Spl** of G^*, which is also regarded as an F-splitting of G_{sc}. The first term in the transfer factor is

$$\Delta_{\mathrm{I}}(\gamma_H, \gamma_G) = \langle \lambda_{T_{\mathrm{sc}}}, \mathbf{s}_T \rangle,$$

where $\lambda_{T_{\mathrm{sc}}}$ is computed using **Spl**, and the pairing is Tate-Nakayama duality.

Lemma 6.2. *For any two pairs (γ_H, γ_G) and (Γ'_H, γ'_G), their quotient*

$$\Delta_{\mathrm{I}}(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G) \coloneqq \Delta_{\mathrm{I}}(\gamma_H, \gamma_G)/\Delta_{\mathrm{I}}(\gamma'_H, \gamma'_G)$$

is independent of **Spl***.*

6.6. It makes sense to regard $R(H, T_H)$ as a Γ_F-stable subset of $R(G^*, T)$ via (the construction of) admissible embedding $T_H \to T$. With this note, the second term is

$$\Delta_{\mathrm{II}}(\gamma_H, \gamma_G) = \prod_{[\alpha] \in [R(G^*, T) - R(H, T_H)]/\Gamma_F} \chi_\alpha\left(\frac{\alpha(\gamma) - 1}{a_\alpha}\right),$$

which can be verified to be well defined. We also define

$$\Delta_{\mathrm{II}}(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G) \coloneqq \Delta_{\mathrm{II}}(\gamma_H, \gamma_G)/\Delta_{\mathrm{II}}(\gamma'_H, \gamma'_G).$$

6.7. For the third term we first deal with when $G = G^*$, and $\psi = \mathrm{id}$. Then we can find $h \in G_{\mathrm{sc}}$ such that $h\gamma_G h^{-1} = \gamma$, and the cohomology class of cocycle $v\colon \sigma \mapsto h\sigma(h)^{-1}$ in $\mathrm{H}^1(F, T_{\mathrm{sc}})$ is independent of h. We use $\mathrm{inv}(\gamma_H, \gamma_G)$ for this class. Then the first part of the third term is

$$\Delta_{\mathrm{III}_1}(\gamma_H, \gamma_G) = \langle \mathrm{inv}(\gamma_H, \gamma_G), \mathbf{s}_T \rangle^{-1},$$

and

$$\Delta_{\mathrm{III}_1}(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G) \coloneqq \Delta_{\mathrm{III}_1}(\gamma_H, \gamma_G)/\Delta_{\mathrm{III}_1}(\gamma'_H, \gamma'_G).$$

In general case where G is not necessarily quasi-split, one cannot define Δ_{III_1} for a pair (γ_H, γ_G) only, and has to define the relative term to another pair (γ'_H, γ'_G), as follows. Let $u(\sigma) \in G_{\mathrm{sc}}$ be such that $\psi\sigma(\psi)^{-1} = \mathrm{Int}\, u(\sigma)$ for $\sigma \in \Gamma_F$, and find $h, h' \in G_{\mathrm{sc}}$ such that

$$h\psi(\gamma_G)h^{-1} = \gamma,$$
$$h'\psi(\gamma'_G)h'^{,-1} = \gamma',$$

and set

$$v(\sigma) = hu(\sigma)\sigma(h)^{-1},$$
$$v'(\sigma) = h'u(\sigma)\sigma(h')^{-1},$$

well-defined up to coboundaries. Since $\partial u = \partial v = \partial v'$, all of which taking values in Z_{sc}, if we let U to be the torus

$$T_{\mathrm{sc}} \times T'_{\mathrm{sc}}/\{(z, z^{-1}) \mid z \in Z_{\mathrm{sc}}\},$$

then (v, v'^{-1}) induces a well-defined class independent of u, h and h'

$$\mathrm{inv}\left(\frac{\gamma_H, \gamma_G}{\gamma'_H, \gamma'_G}\right) \in \mathrm{H}^1(F, U).$$

Note that our notation here is the reciprocal of that in Langlands-Shelstad, because I want to be more consistent in notations with quasi-split case.

On the other hand, we have simply-connected cover $\check{G}_{\mathrm{sc}}$ of the derived group of $\check{G}$, and $\mathcal{T}_{\mathrm{sc}}$ the preimage of $\mathcal{T}$. Let $\tilde{s} \in \mathcal{T}_{\mathrm{sc}}$ be an element that has the same image as s in $\mathcal{T}_{\mathrm{ad}}$, then the isomorphism $\mathcal{T} \to \check{T}$ constructed on the way of choosing an admissible embedding (again, choice of B_H doesn't matter) induces an isomorphism $\mathcal{T}_{\mathrm{sc}} \to \check{T}_{\mathrm{sc}}$, where the latter is the dual torus of $T_{\mathrm{ad}} = T/Z(G)$. The image of $\tilde{s}$ in $\check{T}_{\mathrm{sc}}$ is denoted by $\tilde{s}_T$. Similarly we have $\tilde{s}'_T \in \check{T}'_{\mathrm{sc}}$. They both depends only on the admissible embeddings $T_H \to T$ and $T'_H \to T'$ (after fixing choices on the dual side at the beginning anyway).

The dual torus of U may be canonically identified with

$$\check{U} \simeq \check{T}_{\mathrm{sc}} \times \check{T}'_{\mathrm{sc}}/\{(z, z) \mid z \in Z(\check{G}_{\mathrm{sc}})\}.$$

Let s_U be the image of $(\tilde{s}_T, \tilde{s}'_T)$ in $\check{U}$, then it is independent of choice of $\tilde{s}$. Then s_U is also Γ_F-invariant, hence defines an element $\mathbf{s}_U \in \pi_0(\check{U}^{\Gamma_F})$. Then Tate-Nakayama duality enables us to define

$$\Delta_{III_1}(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G) = \left\langle \mathrm{inv}\left(\frac{\gamma_H, \gamma_G}{\gamma'_H, \gamma'_G}\right), \mathbf{s}_U \right\rangle^{-1}.$$

This is consistent with quasi-split case.

6.8. Continuing with the third term. It is the only part $\mathscr{H} = {}^LH$ will be used. Here we need to use the choice of B_H and B explicitly, and it has no effect on the end product. Such choices together with the χ-datum gives us admissible embeddings

$$\xi_{T_H} \colon {}^LT_H \longrightarrow {}^LH,$$
$$\xi_T \colon {}^LT \longrightarrow {}^LG.$$

Thus we obtain a 1-cocycle $a \colon W_F \to \mathscr{T}$ (with the W_F-action on $\check{T}$ transported to $\mathscr{T}$ via embedding ξ_T, instead of the "original" one), inflated to ${}^LT$ such that

$$\xi \circ \xi_{T_H} = a\xi_T.$$

Its class $\mathbf{a} \in \mathrm{H}^1(W_F, \check{T})$ is independent of the choices of B_H, B, nor splittings on $\check{H}$ or $\check{G}$. Then we define

$$\Delta_{\mathrm{III}_2}(\gamma_H, \gamma_G) = \langle \mathbf{a}, \gamma \rangle,$$

where the pairing is the canonical isomorphism

$$\mathrm{H}^1(W_F, \check{T}) \simeq \mathrm{Hom}_{\mathrm{cont}}(T(F), \mathbb{C}^\times).$$

We also define as before

$$\Delta_{\mathrm{III}_2}(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G) \coloneqq \Delta_{\mathrm{III}_2}(\gamma_H, \gamma_G)/\Delta_{\mathrm{III}_2}(\gamma'_H, \gamma'_G).$$

6.9. The final term of transfer factor is essentially just the discriminant function. For $\gamma \in T(F)$, we define

$$D_{G^*}(\gamma) = \prod_{\alpha \in R(G^*, T)} |\alpha(\gamma) - 1|_F^{\frac{1}{2}}.$$

Similarly we can define $D_H(\gamma_H)$. Then

$$\Delta_{\mathrm{IV}}(\gamma_H, \gamma_G) = D_{G^*}(\gamma) D_H(\gamma_H)^{-1}.$$

Again we let

$$\Delta_{\mathrm{IV}}(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G) \coloneqq \Delta_{\mathrm{IV}}(\gamma_H, \gamma_G)/\Delta_{\mathrm{IV}}(\gamma'_H, \gamma'_G).$$

6.10. Finally, we can define the *relative transfer factor*

$$\Delta(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G) \coloneqq (\Delta_{\mathrm{I}}\Delta_{\mathrm{II}}\Delta_{\mathrm{III}_1}\Delta_{\mathrm{III}_2}\Delta_{\mathrm{IV}})(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G).$$

If G is quasi-split, then we define the *absolute transfer factor*

$$\Delta(\gamma_H, \gamma_G) = \Delta_0(\gamma_H, \gamma_G) \coloneqq (\Delta_{\mathrm{I}}\Delta_{\mathrm{II}}\Delta_{\mathrm{III}_1}\Delta_{\mathrm{III}_2}\Delta_{\mathrm{IV}})(\gamma_H, \gamma_G).$$

In general we have to fix a pair (γ'_H, γ'_G) and define $\Delta(\gamma'_H, \gamma'_G)$ arbitrarily (but nonzero), then define

$$\Delta(\gamma_H, \gamma_G) = \Delta(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G)\Delta(\gamma'_H, \gamma'_G).$$

Theorem 6.3. *The transfer factor $\Delta(\gamma_H, \gamma_G)$ is independent of choice of admissible embedding $T_H \to T$, a-data, or χ-data.*

7. Loose Ends

I didn't include all the properties of transfer factors, how they patch together globally, or how they extend to non-strongly G-regular elements.

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