TOPIC PROPOSAL: ZETA INTEGRALS AND PRINCIPAL L-FUNCTIONS OF GENERAL LINEAR GROUPS

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1. Introduction

One of the goals of the Langlands program is relating Galois representations of local or global fields to (reasonable) representations of algebraic groups over local fields or adeles. One of the most important cases is where $G = \mathrm{GL}(n)$. The local Langlands correspondence gives a unique bijection

$$\left\{ \begin{array}{c} \text{irreducible admissible} \\ \text{representations of } G_F \end{array} \right\} / \sim \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{semisimple } n\text{-dimensional} \\ \text{representations} \\ WD(F) \to {}^L G = \mathrm{GL}(n, \mathbb{C}) \end{array} \right\} / \sim, \tag{1.0.1}$$

where F is a local field, $WD(F)$ is the Weil-Deligne group of F, and ${}^L G$ is the L-group of G. Conjecturally, there is an analogous global statement where the left-hand side is automorphic representations, and the right-hand side is Galois representations. More generally, G can be a reductive group and ${}^L G$ may not be the general linear group, so in order to make the homomorphisms on the right-hand side representations, one needs to fix a representation ρ_0 of ${}^L G$. In the case of $\mathrm{GL}(n)$, ρ_0 can be viewed as chosen to be the standard representation of $\mathrm{GL}(n, \mathbb{C})$. The bijection (1.0.1) should further have the property that the corresponding pair give the same local or global L-functions (although for most general cases, what the L-function on the right-hand side should be seems unclear). In the $\mathrm{GL}(n)$ case, the right-hand side gives Artin L-functions, and Langlands correspondence allows us to study them analytically. The L-functions on the left-hand side will be called principal or standard L-functions. When $n = 1$, it is the well-known case where the left-hand side is the theory of Hecke and Tate, while the right-hand side is the class field theory.

This expository paper, which serves as the topic proposal to fulfill a requirement of my PhD program at the University of Chicago, will discuss the theory of principal L-functions using the method of zeta integrals, including relavant theory of representations, the analytic continuations and the functional equations. I mainly follow [GJ72] and [Jac79] with several other supplements. Both local and global cases will be thoroughly discussed along with a few simple examples (mostly about $\mathrm{GL}(2)$).

2. Local Theory: Nonarchimedean Case

2.1. **Definitions and statement of the result.** Let F be a nonarchimedean local field with valuation ring R_F, $G = G_n = \mathrm{GL}(n, F)$, and $K = \mathrm{GL}(n, R_F)$, a maximal compact subgroup of G.

Definition 2.1.1. Suppose π be a representation of G on a complex vector space V.

(1) Any vector $v \in V$ is called *smooth* if its stablizer is an open subgroup of G.
(2) π is called *smooth* if all the vectors in V are smooth.

(3) π is called *admissible* if its smooth and for any compact open subgroup $H < G$, the subspace of H-fixed vectors is finite dimensional.

Given a smooth representation π of G, we define the contragredient representation $\tilde{\pi}$ to be that on the subspace of smooth vectors in the algebraic dual space, with the G-invariant pairing denoted by $\langle \cdot, \cdot \rangle$. It is easy to see if π is admissible or irreducible, then so is $\tilde{\pi}$ respectively, and $\tilde{\tilde{\pi}} \cong \pi$. A (matrix) coefficient f is a linear combination of functions on G of the form $g \mapsto \langle \pi(g)v, \tilde{v} \rangle$ for some $v \in V$ and $\tilde{v} \in \tilde{V}$. If f is a coefficient of π, then $\check{f}(g) := f(g^{-1})$ is a coefficient of $\tilde{\pi}$.

Denote $\mathcal{M} = \mathcal{M}_n = \mathrm{Mat}(n, F)$ the space of $n \times n$ matrices, and $\mathscr{S}(\mathcal{M})$ the space of Schwartz-Bruhat functions on $\mathcal{M}$. Fix a nontrivial character ψ of F, the Fourier transform of $\Phi \in \mathscr{S}(\mathcal{M})$ is

$$\hat{\Phi}(x) = \int_{\mathcal{M}} \Phi(y)\psi(\mathrm{tr}(yx))\mathrm{d}y,$$

where $\mathrm{d}y$ is the self-dual Haar measure on $\mathcal{M}$. Let $\mathcal{H}(G)$ be the Hecke algebra on G, the algebra of $\mathbb{C}$-valued compactly supported and locally constant functions on G. We also treat G-representations as $\mathcal{H}(G)$-modules.

The main object of our interest is the zeta function $Z(\Phi, s, f)$ defined by

$$Z(\Phi, s, f) = \int_G \Phi(g)|\det g|^s f(g)\mathrm{d}^\times g, \tag{2.1.1}$$

where $\Phi \in \mathscr{S}(\mathcal{M})$, $s \in \mathbb{C}$, f is a coefficient of π and $\mathrm{d}^\times g$ denotes a fixed Haar measure on G.

We can now state the main result for this section.

Theorem 2.1.2. *Suppose π is an irreducible admissible representation of G, then with the notations as above, we have*

(1) *There exists a real number s_0, such that the integral $Z(\Phi, s, f)$ converges absolutely for $\Re(s) > s_0$ and for all Φ and all coefficients f, where $\Re(s)$ denotes the real part of s.*
(2) *Suppose the residue field of F has q elements, then the $\mathbb{C}$-vector space spanned by all $Z(\Phi, s + \frac{1}{2}(n-1), f)$ is a (necessarily principal) fractional ideal in $\mathbb{C}[q^{-s}, q^s]$, denoted by $I(\pi, s)$.*
(3) *There exists an Euler factor $L(s, \pi) = P(q^{-s})^{-1}$ for some polynomial P with $P(0) = 1$, that $I(\pi, s) = L(s, \pi)\mathbb{C}[q^{-s}, q^s]$.*
(4) *There is a rational function $\gamma(s, \pi, \psi)$ in q^{-s} such that for all Φ and all coefficients f,*

$$Z(\hat{\Phi}, 1 - s + \frac{1}{2}(n-1), \check{f}) = \gamma(s, \pi, \psi)Z(\Phi, s + \frac{1}{2}(n-1), f). \tag{2.1.2}$$

2.2. **Some interpretations.** Before we give the proof to Theorem 2.1.2, we give an intuitive idea on the form of the functional equation as well as some useful facts derived from the theorem.

By changing variable $s + \frac{1}{2}(n-1)$ to s, and for simplicity let $\gamma_1(s) = \gamma(s - \frac{1}{2}(n-1), \pi, \psi)$, we can rewrite (2.1.2) into

$$Z(\hat{\Phi}, n - s, \check{f}) = \gamma_1(s)Z(\Phi, s, f).$$

We can embed G into $\mathcal{M}$ since F is a field. Assume for now Φ and $\hat{\Phi}$ have support contained in G, and all integrals converge absolutely, then $Z(\hat{\Phi}, n - s, \check{f})$ can be interpreted as the following convolution on G

$$Z(\hat{\Phi}, n - s, \check{f}) = [J * \Phi * \check{f}_s](e),$$

where $J(g) = \psi(\mathrm{tr}(g^{-1}))|\det g|^{-n}$, $\check{f}_s(g) = |\det g|^{-s}\check{f}(g)$ and e is the identity of G. Note $\check{f}_s$ is a coefficient of $\det^{-s} \otimes \tilde{\pi}$. Since J is a class function on G, we have $J * \Phi = \Phi * J$ and $J * \check{f}_s = \mathrm{constant} \cdot \check{f}_s$ by Schur lemma (see [Car79] or [Bum97]). Hence we get the equation

$$[J * \Phi * \check{f}_s](e) = \mathrm{constant} \cdot [\Phi * \check{f}_s](e) = \mathrm{constant} \cdot Z(\Phi, s, f). \tag{2.2.1}$$

So we see the rank n in functional equation comes from the scaling factor in $\mathrm{d}^\times g = |\det g|^{-n}\mathrm{d}g$. To extend this to the whole $\mathscr{S}(\mathcal{M})$, one uses a version of Plancharel's formula.

An alternative way to describe the functional equation is through bilinear forms and finite rank linear operators.

Definition 2.2.1. An element $\xi \in \mathcal{H}(G)$ is called an *elementary idempotent* if it is of the form

$$\xi(g) = \begin{cases} \sum_{i=1}^m \operatorname{tr}(\rho_i(g^{-1})) & g \in K \\ 0 & g \notin K \end{cases},$$

where $\rho_1, \dots, \rho_m$ are inequivalent irreducible representations of K.

Lemma 2.2.2. *For any function Φ on $\mathcal{M}$ that is K-finite on the right (resp. on the left), there is an elementary idempotent $\xi \in \mathcal{H}(G)$ such that $\Phi * \xi = \Phi$ (resp. $\xi * \Phi = \Phi$). In particular, for any $\Phi \in \mathscr{S}(\mathcal{M})$, there is an elementary idempotent $\xi \in \mathcal{H}(G)$ so that*

$$\xi * \Phi = \Phi * \xi = \Phi.$$

Proof. The first assertion is just elementary representation theory for compact groups. The second one follows from that Φ is K-finite. □

Given $\Phi \in \mathscr{S}(\mathcal{M})$ and $s \in \mathbb{C}$ with $\Re(s) \gg 0$, we have a bilinear form B on $V \times \tilde{V}$ sending $(v, \tilde{v})$ to $Z(\Phi, s, f)$ where $f(g) = \langle \pi(g)v, \tilde{v} \rangle$. By Lemma 2.2.2, there is an elementary idempotent ξ so that $B(\pi(\xi)v, \tilde{\pi}(\check{\xi})\tilde{v}) = B(v, \tilde{v})$ for all v and $\tilde{v}$. Then since $\pi(\xi)$ and $\tilde{\pi}(\check{\xi})$ have finite rank, one can represent B by a linear operator on V, which enables the following definition.

Definition 2.2.3. Given $\Phi \in \mathscr{S}(\mathcal{M})$ and $s \in \mathbb{C}$ with $\Re(s) \gg 0$, the operator $Z(\Phi, s, \pi)$ on V is defined by

$$\langle Z(\Phi, s, \pi)v, \tilde{v} \rangle = Z(\Phi, s, f)$$

for all $v \in V$ and $\tilde{v} \in \tilde{V}$, where $f(g) = \langle \pi(g)v, \tilde{v} \rangle$. We also define $Z(\Phi, s, \tilde{\pi})$ by

$$\langle Z(\Phi, s, \tilde{\pi})v, \tilde{v} \rangle = \langle v, Z(\Phi, s, \tilde{\pi})\tilde{v} \rangle.$$

The functional equation is then simply a statement that up to a constant, $Z(\hat{\Phi}, 1 - s + \frac{1}{2}(n-1), \tilde{\pi})$ is the adjoint operator of $Z(\Phi, s + \frac{1}{2}(n-1), \pi)$.

We list here some important facts [Jac79].

(1) Suppose π is not irreducible but central, i.e. there exists a quasicharacter $\omega : F^\times \to \mathbb{C}^\times$ that $\pi(a) = \omega(a)1_V$ for $a \in F^\times$, then the statement of Theorem 2.1.2 makes sense for π (although may fail to be true).
(2) If Theorem 2.1.2 is true for an admissible central representation π, then it is true for its irreducible components.
(3) The ϵ-factor is defined as

$$\epsilon(s, \pi, \psi) = \gamma(s, \pi, \psi) \frac{L(s, \pi)}{L(1 - s, \tilde{\pi})}. \tag{2.2.2}$$

It is necessarily a monomial in q^{-s}.

2.3. **Reduction to supercuspidal representations.** The proof of Theorem 2.1.2 can be divided into two steps.

(1) Reduce the problem to the case where the representation π is supercuspidal (Definition 2.3.6).
(2) Prove the theorem for supercuspidals, which enjoy the property that coefficients of π are compactly supported modulo the center of G. One then reduces the problem to the already-proved $n = 1$ case [Tat67].

This subsection is dedicated to the first step. Let P be the standard parabolic subgroup of type $(n_1, \dots, n_r)$ of G, whose elements are those in the form

$$p = \begin{pmatrix} g_1 & & u_{ij} \\ & \ddots & \\ 0 & & g_r \end{pmatrix}, \quad g_i \in \mathrm{GL}(n_i). \tag{2.3.1}$$

Let $N = N(P)$ be the unipotent radical of P, whose elements are those with $g_i = e_{n_i}$. We then have $P/N \cong \prod_i G_{n_i}$. Suppose σ_i is an admissible representation of G_{n_i}, then $\sigma = \boxtimes_i \sigma_i$ can be viewed as an admissible representation of P trivial on N.

Definition 2.3.1. Suppose (σ, W) is a representation of P trivial on N. The *induced representation* $(\pi, V) = \mathrm{Ind}_P^G \sigma$ of G is defined on the space of functions $f : G \to W$ satisfying conditions

(1) For $g \in G$, $p \in P$,
$$f(pg) = \delta_P(p)^{\frac{1}{2}} \sigma(p) f(g),$$
where δ_P is the modular quasicharacter with respect to a fixed left Haar measure on P.

(2) There is an open subgroup $G' < G$ such that for all $g \in G$, $h \in G'$,
$$f(gh) = f(g).$$

The action π is the right translation $(\pi(g)f)(g') = f(g'g)$.

Proposition 2.3.2. *If (σ, W) is admissible, so is $\mathrm{Ind}_P^G \sigma$ and $\mathrm{Ind}_P^G \tilde{\sigma}$. Moreover, we have $(\mathrm{Ind}_P^G \sigma)^\sim \cong \mathrm{Ind}_P^G \tilde{\sigma}$.*

Proof. For the first assertion see Theorem 2.9(i) of [GJ72]. It is completely elementary once one notices for any compact open subgroup $G' < G$, the double cosets $P \backslash G / G'$ is a finite set. We prove the second assertion. Suppose h is a continuous function on G such that $h(pg) = \delta_P(p) h(g)$, the Iwasawa decomposition $G = PK$ shows h is completely determined on K. The linear form $h \mapsto \int_K h(k) \mathrm{d}k$ is invariant under right translation by G (Lemma 2.6.1 of [Bum97]). Denote this form by $\int_{P \backslash G} \mathrm{d}g$, and let V, $\tilde{V}$ be the respective spaces of $\mathrm{Ind}_P^G \sigma$ and $\mathrm{Ind}_P^G \tilde{\sigma}$. One then has a nondegenerate G-invariant pairing on $V \times \tilde{V}$ by
$$\langle f, \tilde{f} \rangle = \int_{P \backslash G} \langle f(g), \tilde{f}(g) \rangle_{W \times \tilde{W}} \mathrm{d}g,$$
hence the identification $(\mathrm{Ind}_P^G \sigma)^\sim \cong \mathrm{Ind}_P^G \tilde{\sigma}$. □

Proposition 2.3.2 allows us to talk about Theorem 2.1.2 for induced representations.

Proposition 2.3.3. *With the notations in Definition 2.3.1, suppose each σ_i admits a central quasicharacter, and Theorem 2.1.2 is true for each σ_i, then it is true for $\pi = \mathrm{Ind}_P^G \sigma$. Moreover, we have*
$$I(\pi, s) = \prod_i I(\sigma_i, s),$$
$$\gamma(s, \pi, \psi) = \prod_i \gamma(s, \sigma_i, \psi).$$

Proof (sketch). Section 2.2 of [Jac79] gives a clean account for this proposition. The proof is basically manipulate the integrals so that $Z(\Phi, s + \frac{1}{2}(n-1), f)$ can be written as a finite linear combination of products $\prod_i Z(\Phi_i, s + \frac{1}{2}(n_i - 1), f_i)$. Running the argument backwards shows any product $\prod_i Z(\Phi_i, s + \frac{1}{2}(n_i - 1), f_i)$ equals $Z(\Phi, s + \frac{1}{2}(n-1), f)$ for some Φ and f. A careful comparison of the linear coefficients of $Z(\Phi, s + \frac{1}{2}(n-1), f)$ when written as combination of products with those of $Z(\hat{\Phi}, 1 - s + \frac{1}{2}(n-1), \check{f})$ concludes the proof. □

We need some observations (Section 2.2 of [Jac79]) to make a relatively clean proof. Let V be the vector space of π, and W the space of σ, then for $v \in V, \tilde{v} \in \tilde{V}$, the function $H : G \times G \to \mathbb{C}$ defined by $H(g, g') = \langle v(g), \tilde{v}(g') \rangle_{W \times \tilde{W}}$ satisfies:

(1) $H(pg, p'g') = \delta_P(p) H(g, g')$ for $g, g' \in G$ and $p/N = p'/N \in P/N$.

(2) For any $g, g' \in G$, the function $p \mapsto H(pg, g')$ is a coefficient of $\delta_P^{\frac{1}{2}} \otimes \sigma$.

(3) H is $K \times K$-finite on the right.

Then the coefficient f is by definition
$$f(g) = \langle \pi(g) v, \tilde{v} \rangle = \int_K H(kg, k) \mathrm{d}k. \tag{2.3.2}$$
Conversely, for any function H satisfying the three conditions above, the function f defined by (2.3.2) is necessarily a coefficient of π. The coefficient $\check{f}$ is similarly defined by
$$\check{f}(g) = \int_K \tilde{H}(kg, k) \mathrm{d}k,$$
where $\tilde{H}(g, g') = H(g', g)$.

Proof of Proposition 2.3.3. Write (2.1.1) (with complex parameter $s+\frac{1}{2}(n-1)$) into

$$\begin{aligned} Z(\Phi, s+\frac{1}{2}(n-1), f) &= \int_G \Phi(g)|\det g|^{s+\frac{1}{2}(n-1)} \left(\int_K H(kg,k)\mathrm{d}k\right) \mathrm{d}^\times g \\ &= \iint_{K\times K} \mathrm{d}k\mathrm{d}k' \int_P \Phi(k^{-1}pk')|\det p|^{s+\frac{1}{2}(n-1)} H(pk',k)\mathrm{d}_l p. \end{aligned} \tag{2.3.3}$$

Break the integral over P into integrals over N and $P/N = \prod_i G_{n_i}$, and write

$$\Psi_\Phi(g_i;k,k') = \Psi_\Phi(g_1,\dots,g_r;k,k') = \int_N \Phi(k^{-1}pk')\mathrm{d}u = \int_N \Phi(k^{-1}pk')\prod_{i,j}\mathrm{d}u_{ij},$$

$$h_\sigma(g_i;k,k') = h_\sigma(g_1,\dots,g_r;k,k') = H(pk',k)\delta_P^{-\frac{1}{2}}(p),$$

where u_{ij} are those in (2.3.1) and note $\prod_{i,j}\mathrm{d}u_{ij}$ is the Haar measure on N. Then (2.3.3) becomes

$$\iint_{K\times K} \mathrm{d}k\mathrm{d}k' \int_{P/N} \Psi_\Phi(g_i;k,k')h_\sigma(g_i;k,k')\prod_i |\det g_i|^{s+\frac{1}{2}(n_i-1)}\prod_i \mathrm{d}^\times g_i. \tag{2.3.4}$$

We have then, by K-finiteness of Φ

$$\Phi(k^{-1}pk') = \sum_{\alpha,\beta} c_\alpha(k)d_\beta(k')\Phi_{\alpha\beta}(p),$$

where α,β runs over some finite index set, and c_α and d_β are families of functions (depending on Φ), and $\Phi_{\alpha\beta}$ is a basis for the vector space spanned by K-translations of Φ. Thus we have

$$\Psi_\Phi(g_i;k,k') = \sum_{\alpha,\beta} c_\alpha(k)d_\beta(k')\Psi_{\Phi_{\alpha\beta}}(g_i;e,e). \tag{2.3.5}$$

Similar result holds for h_σ as well. So after integrating over $K\times K$, one sees $Z(\Phi, s+\frac{1}{2}(n-1), f)$ is a finite linear combination of products $\prod_i Z(\Phi_i, s+\frac{1}{2}(n_i-1), f_i)$, for some $\Phi_i \in \mathscr{S}(\mathcal{M}_{n_i})$ and f_i coefficient of σ_i. Hence we proved the convergence for $\Re(s) \gg 0$ by induction hypothesis, as well as the inclusion $I(\pi,s)\subset \prod_i I(\sigma_i)$.

Conversely, given zeta integrals $Z(\Phi_i, s+\frac{1}{2}(n_i-1), f_i)$, first note it is easy to find $\Phi\in\mathscr{S}(\mathcal{M}_n)$ that $\int_N \Phi(p)\mathrm{d}u = \prod_i \Phi_i(g_i)$. Then by Lemma 2.2.2, we can find K-finite functions ξ_1 and ξ_2 that

$$\iint_{K\times K} \Phi(k^{-1}gk')\xi_1(k)\xi_2(k')\mathrm{d}k\mathrm{d}k' = \Phi(g).$$

One then just (mostly) reverse the calculation above to get

$$Z(\Phi, s+\frac{1}{2}(n-1), f) = \prod_i Z(\Phi_i, s+\frac{1}{2}(n_i-1), f_i)$$

for some coefficient f, so that $I(\pi,s) = \prod_i I(\sigma_i)$.

So far we proved the first three assertions in Theorem 2.1.2 for π. Now we prove the functional equation. Lemma 3.4.0 of [GJ72] shows

$$\Psi_{\hat\Phi}(g_i;k,k') = \hat\Psi_\Phi(g_i;k',k),$$

where $\hat\Psi$ denotes the Fourier transform of Ψ on $\prod_i \mathcal{M}_{n_i}$. One also easily shows

$$h_{\check\sigma}(g_i;k,k') = h_\sigma(g_i^{-1};k',k),$$

so we have

$$Z(\hat\Phi, s+\frac{1}{2}(n-1), \check f) = \iint_{K\times K}\mathrm{d}k\mathrm{d}k' \int_{P/N} \hat\Psi_\Phi(g_i;k',k)h_\sigma(g_i^{-1};k',k)\prod_i|\det g_i|^{s+\frac{1}{2}(n_i-1)}\prod_i \mathrm{d}^\times g_i. \tag{2.3.6}$$

We have a similar result as in (2.3.5) for the contragredient side. So we have, after integrating those linear coefficients (c_α and d_β) over $K\times K$, (2.3.4) has exactly same linear coefficients as does (2.3.6) when written into linear combination of products $\prod_i Z(\Phi_i, s+\frac{1}{2}(n_i-1), f_i)$ and $\prod_i Z(\hat\Phi_i, s+\frac{1}{2}(n_i-1), \check f_i)$ respectively. So the induction hypothesis gives the functional equation for π, and also the product formula for the γ-factors. □

In order to reduce Theorem 2.1.2 to supercuspidals (Definition 2.3.6), it suffices to show that any irreducible admissible representation can be realized as a component of the induced representation of some supercuspidal representation.

If we have parabolic subgroups P and P' of G such that $P' < P$, then $N' = N(P') > N = N(P)$. Let σ' be an admissible representation of P'/N', and $\sigma = \mathrm{Ind}_{P'/N}^{P/N}\sigma'$, it is easily shown the natural identification (equation (2.10) of [GJ72])

$$\mathrm{Ind}_{P'}^{G}\sigma' \cong \mathrm{Ind}_{P}^{G}\sigma. \tag{2.3.7}$$

Now let (π, V) be an irreducible admissible representation of G, P any proper parabolic subgroup of G and N its unipotent radical, denote by $V(N)$ the subspace spanned by vectors of the form $v - \pi(u)v$ for all $u \in N$. We have immediately that $V(N)$ is actually a P-invariant subspace since P normalizes N.

Definition 2.3.4. Let $V(N)$ be as above, and $V_N = V/V(N)$, the *Jacquet module* is the quotient representation (π_N, V_N) of P.

Proposition 2.3.5. *Let π, π_N be as above. If π is admissible, so is π_N viewed as a representation of P/N.*

Proof. See [Car79] or [Cas16b]. □

Definition 2.3.6. Suppose $n > 1$. An irreducible admissible representation (π, V) of G is called *supercuspidal* if for all proper parabolic subgroup $P < G$, the Jacquet module $V_{N(P)} = 0$. An irreducible admissible representation $\boxtimes_i \sigma_i$ of $\prod_i G_{n_i}$ is called *supercuspidal* if each σ_i is.

Proposition 2.3.7 (Frobenius reciprocity)**.** *Let π be any admissible representation of G and σ one of P trivial on N, then we have isomorphism by evaluating on the identity $e \in G$*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G\sigma) \cong \mathrm{Hom}_P(\delta_P^{-\frac{1}{2}} \otimes \pi_N, \sigma)$$
$$\varphi \mapsto (w \mapsto \varphi(w)(e)).$$

Proof. It is just standard argument and we omit here. See Proposition 2.12 of [GJ72]. □

Corollary 2.3.8. *If π is irreducible and not supercuspidal, then π can be realized as a subrepresentation of $\mathrm{Ind}_P^G\sigma$, where P a proper parabolic subgroup of G, σ some irreducible admissible representation of P trivial on $N(P)$.*

Proof. Let P be a parabolic subgroup that $V_N \neq 0$ where $N = N(P)$. Combining Frobenius reciprocity and the admissibility of π, we can show $\sigma = \delta_P^{-\frac{1}{2}} \otimes \bar{\pi}_N$, the twisted quotient of π_N by its largest proper submodule. Proposition 2.3.5 ensures that σ is admissible. □

Theorem 2.3.9. *Any irreducible admissible representation π of G can be realized as a subrepresentation of a representation of the form $\mathrm{Ind}_P^G\sigma$, where P is a parabolic subgroup of G and σ a supercuspidal representation of $P/N(P)$.*

Proof. Combining (2.3.7) and Corollary 2.3.8, one proves it by induction on n. □

2.4. **Supercuspidal representations.** We now present the proof of Theorem 2.1.2 in supercuspidal case. Throughout this subsection, (π, V) is a supercuspidal representation of G, and by definition π is irreducible.

Proposition 2.4.1. *Let (π, V) be a supercuspidal representation of G. We have*

1. *The coefficients of π are compactly supported modulo $Z(G)$, the center of G.*
2. *If the central quasicharacter ω of π is actually a character, then π is preunitary, i.e. there exists a G-invariant Hermitian inner product on V.*
3. *The contragredient $\tilde{\pi}$ is also supercuspidal.*

Proof. Let coefficient $f(g) = \langle \pi(g)v, \tilde{v} \rangle$. We have the Cartan decomposition $G = KA^-K$, where A^- is the subset of diagonal matrices $a = \mathrm{diag}(a_1, \dots, a_n)$ that $|a_i| \leq |a_{i+1}|$ for all $1 \leq i \leq n-1$. Since f is K-finite on both sides, it suffices to prove f is compactly supported on $A^-/F^\times$, i.e. there is $c > 0$ such that for any $a \in A^-$, $|a_i/a_{i+1}| < c$ for some i implies $f(a) = 0$. Since there are finitely many i, it suffices to show the existence of c for a fixed i. Now let P be the parabolic subgroup of type $(i, n-i)$, $N = N(P)$, then π is supercuspidal implies we can find compact open subgroup $U_1 < N$ such that the average of v over U_1

vanishes. Let $U_2 < N$ be a compact open subgroup that fixes $\tilde{v}$. Then the nondiagonal matrix entries of U_1 are bounded above by some c_1, and those of U_2 are bounded below by some $c_2 > 0$, if we let $c = c_2/c_1$, then a simple calculation shows if $a \in A^-$ and $|a_i/a_{i+1}| < c$, then $aU_1a^{-1} \subset U_2$, hence

$$f(a) = \int_{U_1} \langle \pi(a)v, \tilde{v} \rangle \mathrm{d}u = \int_{U_1} \langle \pi(a)v, \tilde{\pi}(aua^{-1})\tilde{v} \rangle \mathrm{d}u = \int_{U_1} \langle \pi(a)\pi(u^{-1})v, \tilde{v} \rangle \mathrm{d}u = 0,$$

the final equality due to that averaging over U_1 kills v. So the first assertion is proved. If ω is a character, the inner product on V can be defined as

$$(v_1, v_2) = \int_{G/Z} \langle \pi(g)v_1, \tilde{v} \rangle \overline{\langle \pi(g)v_2, \tilde{v} \rangle} \mathrm{d}^\times g,$$

where $\tilde{v}$ is a fixed nonzero element of $\tilde{V}$. It is well defined by the first assertion, and positive definite by the irreducibility of $\tilde{\pi}$. Firnally, since twisting by a quasicharacter of G does not affect supercuspidality, we may assume π is preunitary, hence $\tilde{\pi}$ may be identified with the complex conjugate of π, which is supercuspidal. □

Definition 2.4.2. We define a subspace $\mathscr{S}_0(\mathcal{M})$ of $\mathscr{S}(\mathcal{M})$ as the functions $\Phi \in \mathscr{S}(\mathcal{M})$ satisfying

(1) Φ vanishes on singular matirces, i.e. it is supported in G.
(2) For $N = N(P)$ where P any parabolic subgroup of G, and $g_1, g_2 \in G$, we have

$$\int_N \Phi(g_1 u g_2) \mathrm{d}u = 0.$$

Lemma 2.4.3. *$\mathscr{S}_0(\mathcal{M})$ is stable under Fourier transform.*

Proof. Calculation. See [GJ72], Lemma 5.3. □

Proposition 2.4.4. *Let $Z(\Phi, s, \pi)$ be the operator associated to the zeta function, then*

(1) *$Z(\Phi, s, \pi)$ can be identified with an element in $V \otimes \tilde{V}$.*
(2) *Given any element $T \in V \otimes \tilde{V}$, and $s \in \mathbb{C}$, there is a $\Phi \in \mathscr{S}_0(\mathcal{M})$ such that $Z(\Phi, s, \pi) = T$.*
(3) *Given any nonzero $v \in V$, the set of $u \in V$ that there is $\Phi \in \mathscr{S}_0(\mathcal{M})$, $c \neq 0$ and $n \in \mathbb{Z}$ so that*

$$Z(\Phi, s, \pi) = cq^{-ns}u \quad \textit{for all } s \in \mathbb{C}$$

spans V.

Proof (ideas). The first assertion is obvious. For the third assertion one notices that if u belongs to the prescribed set, then so does $\pi(g)u$ for any $g \in G$, so by irreducibility, it suffices to prove this set is nonempty. For the second assertion, one reduces to Schur orthogonality. For details see [GJ72], Proposition 5.5. □

Remark 2.4.5. The vanishing condition of functions in $\mathscr{S}_0(\mathcal{M})$ is seemingly very restrictive, but Proposition 2.4.4 shows $\mathscr{S}_0(\mathcal{M})$ is still a large enough space of functions to work with.

Proposition 2.4.6 (Plancharel's formula)**.** *Suppose $\Phi \in \mathscr{S}(\mathcal{M})$ and $\Psi \in \mathscr{S}_0(\mathcal{M})$, then for $0 < \Re(s) < n$ the integrals*

$$\iint_{G\times G} \Phi(g)\hat{\Psi}(h) \langle \pi(g)v, \tilde{\pi}(h)\tilde{v} \rangle |\det g|^s |\det h|^{n-s} \mathrm{d}^\times g \mathrm{d}^\times h$$

and

$$\iint_{G\times G} \hat{\Phi}(g)\Psi(h) \langle \pi(g^{-1})v, \tilde{\pi}(h^{-1})\tilde{v} \rangle |\det g|^{n-s} |\det h|^s \mathrm{d}^\times g \mathrm{d}^\times h$$

are absolutely convergent and are equal.

Proof. Calculation. That $\Psi \in \mathscr{S}_0(\mathcal{M})$ enables direct proof of convergence. See [GJ72], Proposition 5.6. □

Proposition 2.4.7. *There is a unique scalar $\gamma(s)$ such that for all $\Phi \in \mathscr{S}_0(\mathcal{M})$*

$$Z(\hat{\Phi}, n - s, \tilde{\pi}) = \gamma(s) Z(\Phi, s, \pi).$$

Proof. This is just our discussion for (2.2.1) plus some argument about convergence. See [GJ72], Proposition 5.8. □

Now we are ready to prove Theorem 2.1.2.

Proof of Theorem 2.1.2 for supercuspidals. Retain all the relevant notations above. By Proposition 2.4.1, there is a compact open subgroup $G' < G$ such that Φ and f are invariant under G', and the support of f is contained in a finite union $\bigcup_i G'Zg_i$. Thus up to a constant,

$$Z(\Phi, s, f) = \sum_i f(g_i)|\det g_i|^s \int_{F^\times} \Phi(ag_i)|a|^{ns}\omega(a)\mathrm{d}^\times a.$$

This is a finite sum of convergent integrals for $\Re(s) > 0$ [Tat67]. Hence we proved the convergence assertion. Moreover, one has an Euler factor $L(ns, \omega)$ for the integral over $F^\times$, and $Z(\Phi, s, f)L(ns, \omega)^{-1}$ is a polynomial in q^{-s} and q^s. Replacing Φ and f by a translation by any element in G, one easily sees the zeta functions form a nonzero fractional ideal of $\mathbb{C}[q^{-s}, q^s]$. So we proved the second assertion. By Proposition 2.4.4, for arbitrary nonzero $\tilde{v} \in \tilde{V}$, we may choose $\Psi \in \mathscr{S}_0(\mathcal{M})$ so that $Z(\hat{\Psi}, n-s, \tilde{\pi})\tilde{v} = q^s\tilde{u}$ for some nonzero $\tilde{u} \in \tilde{V}$. Hence we have $Z(\Psi, s, \tilde{\tilde{\pi}}) = \gamma(s)^{-1}q^s\tilde{u}$. Let $v \in V$ be arbitrary, then Proposition 2.4.6 gives

$$q^s Z(\Phi, s, f) = \gamma(s)^{-1}q^s Z(\hat{\Phi}, n-s, \check{f}),$$

where $f(g) = \langle \pi(g)v, \tilde{u}\rangle$. Since v is arbitrary, and all those $\tilde{u}$ for varying Ψ span $\tilde{V}$, we are done. □

For supercuspidals, we have an even more precise result (Proposition 5.11 of [GJ72]).

Proposition 2.4.8. *Suppose π is supercuspidal and $n > 1$, then $L(s, \pi) = L(s, \tilde{\pi}) = 1$.*

2.5. **Examples.** For more detailed and comprehensive results one can refer to [GJ72] or [Jac79]. Here we give the results for spherical functions, which are indispensable for global theory; we also give results for GL(2) in the principal series case.

If π is an irreducible admissible representation of G, then the subspace V_0 of K-invariant vectors is at most one dimensional.

Definition 2.5.1. With the notations above, an irreducible admissible representation of G is called *spherical* if V_0 is nontrivial.

In general, for any admissible π, assume V_0 is one dimensional, then for the contragredient representation, $\tilde{V}_0$ is one dimensional as well. Choose $v_0 \in V_0$, $\tilde{v}_0 \in \tilde{V}_0$, so that $\langle v_0, \tilde{v}_0\rangle = 1$, then we define the spherical function f_0 attached to π by

$$f_0(g) = \langle \pi(g)v_0, \tilde{v}_0\rangle.$$

Now let P be the subgroup of upper triangular matrice, then $P/N \cong (F^\times)^n$. Let σ_i be n unramified quasicharacters of $F^\times$, and $\sigma = \boxtimes_i\sigma_i$ be a one dimensional representation of P trivial on N, and let π be the induced representation $\mathrm{Ind}_P^G\sigma$. Then it turns out V_0 is one dimensional for this π, and the spherical function attached to π is

$$f_0(g) = \langle \pi(g)\varphi_0, \tilde{\varphi}_0\rangle = \int_K \varphi_0(kg)\tilde{\varphi}_0(k)\mathrm{d}k = \int_K \varphi_0(kg)\mathrm{d}k,$$

where $\varphi_0(pk) = \delta_P(p)^{\frac{1}{2}}\sigma(p)$ and $\tilde{\varphi}_0(pk) = \delta_P(p)^{\frac{1}{2}}\sigma^{-1}(p)$. Let π_0 be the only irreducible component of π that contains the K-invariant subspace, then f_0 is also a spherical function attached to π_0, and moreover we have the following.

Proposition 2.5.2. *With the notations above,*

(1) $L(s, \pi_0) = \prod_i L(s, \sigma_i)$.
(2) $\epsilon(s, \pi_0, \psi) = \prod_i \epsilon(s, \sigma_i, \psi)$, *and equals* 1 *if the exponent of ψ is zero and K has measure one.*

Proof. See [GJ72], Lemma 6.10 and Proposition 6.12. □

When $n = 2$, the only nontrivial standard parabolic subgroup of G is the Borel group B. An irreducible representation of $B/N(B)$ is the same as a quasicharacter $\chi = \chi_1 \boxtimes \chi_2$ where χ_1 and χ_2 are two quasicharacters of $F^\times$. The induced representation $\mathrm{Ind}_B^G\chi$ is often denoted by $\mathcal{B}(\chi_1, \chi_2)$. The following result is well known [Bum97].

Theorem 2.5.3. *For any χ_1 and χ_2,*

(1) *If* $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$*, then* $\pi = \mathcal{B}(\chi_1, \chi_2)$ *is irreducible and* $L(s,\pi) = L(s,\chi_1)L(s,\chi_2)$*. Moreover,* $\mathcal{B}(\chi_1,\chi_2) \cong \mathcal{B}(\chi_2,\chi_1)$.
(2) *If* $\chi_1\chi_2^{-1} = |\cdot|$*, then* $\mathcal{B}(\chi_1,\chi_2)$ *contains a unique irreducible proper invariant subspace* $\mathcal{B}_s$ *and its quotient* $\mathcal{B}_f$ *by this space is one dimensional.*
(3) *If* $\chi_1\chi_2^{-1} = |\cdot|^{-1}$*, then* $\mathcal{B}(\chi_1,\chi_2)$ *contains a unique one dimensional invariant subspace spanned by* $|\det(\cdot)|$ *that is isomorphic to* $\mathcal{B}_f$*, and its quotient by this subspace is isomorphic to* $\mathcal{B}_s$.

3. Local Theory: Archimedean Case

3.1. $(\mathfrak{g}, K)$-modules and the Hecke algebra $\mathcal{H}(G, K)$.

Let F now be an archimedean local field, so $F = \mathbb{R}$ or $F = \mathbb{C}$. Let G again be $\mathrm{GL}(n, F)$, and $K = \mathrm{O}(n)$ if $F = \mathbb{R}$ and $K = \mathrm{U}(n)$ if $F = \mathbb{C}$. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the *real* Lie algebras of G and K respectively, also let $\mathcal{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$.

In archimedean case, finding suitable definitions is actually the hardest part. Instead of dealing with representations of G, one has to move to so-called $(\mathfrak{g}, K)$-modules, or Harish-Chandra modules. A thorough account of basic properties of $(\mathfrak{g}, K)$-modules is too long for this article, so we only give the definition and cite relevant results when needed. An overall convenient reference is a short note by Casselman [Cas16a].

Definition 3.1.1. Let (π, V) be a continuous representation of G on a reasonable (e.g. complete locally convex) topological vector space V.

(1) π is *irreducible* if it contains no *closed* nontrivial subrepresentation of G.
(2) A K*-finite* vector v is such that $\pi(K)v$ span a finite dimensional vector space. Denote by V_0 the subspace of K-finite vectors.
(3) A C^1 vector v is such that the limit
$$\pi(X)v = \lim_{t\to 0} \frac{\pi(\exp(tX))v - v}{t}$$
exists for all $X \in \mathfrak{g}$. Inductively, a C^k vector v is a C^1 vector with $\pi(X)v$ being C^{k-1} for all $X \in \mathfrak{g}$, and a smooth vector is one that is C^k for all $k > 0$. Denote V^∞ the subspace of smooth vectors.
(4) Denote $V_\pi = V_0 \cap V^\infty$.

V^∞ is G-invariant while V_0 is not; however, they are both $\mathfrak{g}$-invariant, and the action of $\mathfrak{g}$ is compatible with the restricted action of K in certain ways. This leads to the definition of $(\mathfrak{g}, K)$-modules, which has the advantage of being purely algebraic.

Definition 3.1.2. A $(\mathfrak{g}, K)$*-module* is a complex vector space V that simultaneously admits a $\mathfrak{g}$-action and a K-action, both denoted by π, satisfying

(1) V consists solely of K-finite vectors, and the action of K is continuous (equivalently, V is a union of continuous finite dimensional K-representations).
(2) The differential of K-action is the same as the $\mathfrak{k}$-action inherited from that of $\mathfrak{g}$.
(3) For any $k \in K$, $X \in \mathfrak{g}$ and $v \in V$
$$\pi(k)\pi(X)\pi(k^{-1})v = \pi(\mathrm{Ad}(k)X)v.$$

For any representation (π, V) of G, we have an associated $(\mathfrak{g}, K)$-module V_π, which is a dense subspace of V.

Definition 3.1.3. A $(\mathfrak{g}, K)$-module is called *admissible* if every irreducible representation of K appears with finite multiplicity. A representation of G is called *admissible* if its associated V_0 has this property.

We can also talk about admissibility of the contragredient representation, as well as of the dual $(\mathfrak{g}, K)$-modules, and we have that they are admissible if and only if the original ones are, since the admissibility is determined by the decomposition as K-representations. A partial converse of the associated $(\mathfrak{g}, K)$-module to a G-representation is the following result (Corollary 4.19 of [Wal79]), which is a corollary of the important Theorem 3.2.4.

Theorem 3.1.4. *Let* W *be an irreducible admissible* $(\mathfrak{g}, K)$*-module, then there exists an irreducible admissible representation* (π, V) *of* G *with* V_π *isomorphic to* W *as* $(\mathfrak{g}, K)$*-modules.*

With Theorem 3.1.4, one can talk about matrix coefficients of an irreducible and admissible $(\mathfrak{g}, K)$-module, but with the concern that it may depend on the choice of the G-representation, which in general is not unique. This concern is eliminated by the following proposition [Bor72].

Proposition 3.1.5. *Let (π, V) be an admissible representation of G. If $v \in V_0$ a K-finite vector, and $\tilde{v}$ any continuous linear functional on V, then the matrix coefficient $g \mapsto \langle \pi(g)v, \tilde{v}\rangle$ is analytic.*

Proposition 3.1.5 implies that any matrix coefficient of a $(\mathfrak{g}, K)$-module depends only on its Taylor expansion, i.e. the action of $\mathcal{U}(\mathfrak{g})$, hence independent of the choice of the G-representation it is associated to. By an abuse of language, we call such function a matrix coefficient *of G*.

Remark 3.1.6. One can actually make the map from the category of $(\mathfrak{g}, K)$-modules to the category of G-representations functorial. However, there are again several inequivalent ways to do this. For a "canonical" construction, see [Cas89a].

An equivalent way to formulate matrix coefficients is through the Hecke algebra. This is the one adopted by [GJ72] and [JL70], although most of the more recent literature seems in favor of $(\mathfrak{g}, K)$-module.

Definition 3.1.7. Let $\mathcal{H}_1(G, K)$ be the subalgebra of K-finite functions (on both sides) in $C_c^\infty(G)$, and $\mathcal{H}_2(G, K)$ be the algebra generated by matrix coefficients of irreducible representations of K, viewed as distributions on G supported in K. Define the *Hecke algebra* $\mathcal{H}(G, K)$ by

$$\mathcal{H}(G, K) = \mathcal{H}_1(G, K) + \mathcal{H}_2(G, K).$$

We also simply use $\mathcal{H}$, $\mathcal{H}_1$, or $\mathcal{H}_2$ if no confusion arises.

Definition 3.1.8. An *admissible representation* of $\mathcal{H}(G, K)$ is a complex representation (π, V) satisfying

(1) For any $v \in V$, it can be written as a finite sum
$$v = \sum_i \pi(f_i) v_i,$$
for some $v_i \in V$ and $f_i \in \mathcal{H}_1(G, K)$.

(2) For any elementary idempotent ξ, $\pi(\xi)V$ is finite dimensional.

(3) For any elementary idempotent ξ and any $v \in \pi(\xi)V$, the map $f \mapsto \pi(f)v$, where $f \in \xi * \mathcal{H}_1 * \xi$, is continuous.

Remark 3.1.9. The topology of $\xi * \mathcal{H}_1 * \xi$ is chosen as the subspace topology of the Schwartz space $\mathscr{S}(G)$, which is a Fréchet space; for details on $\mathscr{S}(G)$ see [Cas89b]. The point is here we have a reasonable topological vector space to work with.

A definition of matrix coefficients "of G" in terms of admissible representation of $\mathcal{H}(G, K)$ can be found in [GJ72]; we do not elaborate here.

Clearly a representation of G will induce both a $(\mathfrak{g}, K)$-module and a representation of $\mathcal{H}(G, K)$. Moreover, the subspace V_π of the $(\mathfrak{g}, K)$-module is invariant under $\mathcal{H}(G, K)$, and by a simple calculation we have for any $v \in V^\infty$, $f \in C_c^\infty(G)$ and $X \in \mathfrak{g}$,

$$\pi(X)\pi(f)v = \pi(L_X f)v, \tag{3.1.1}$$

where L_X denotes the left associated action of X. Conversely, given an admissible representation of $\mathcal{H}(G, K)$, one can find for any vector v an $f \in \mathcal{H}$ that fixes v, hence one can define an action of $\mathfrak{g}$ using (3.1.1), and an action of K using Dirac sequences. One in this way obtains a $(\mathfrak{g}, K)$-module hence a G-representation. There are of course many analytic details necessary for this correspondence, but we do not attempt it here.

3.2. **The result and proof.** As in nonarchimedean case, we denote $\mathscr{S}(\mathcal{M})$ the space of Schwartz functions on the matrix algebra $\mathcal{M} = \mathrm{Mat}(n, \mathbb{R})$ or $\mathrm{Mat}(n, \mathbb{C})$. We also denote $\mathscr{S}_0(\mathcal{M})$ the subspace of functions Φ of the form

$$\Phi(x) = P(x) \exp(-\pi \sum x_{ij}^2) \qquad \text{if } F = \mathbb{R},$$
$$\Phi(z) = P(z, \bar{z}) \exp(-\pi \sum z_{ij}\bar{z}_{ij}) \qquad \text{if } F = \mathbb{C}.$$

Note elements in $\mathscr{S}_0(\mathcal{M})$ are K-finite. Correspondingly, if we choose the additive character of F to be

$$\psi(x) = \exp(-2\pi i x) \qquad \text{if } F = \mathbb{R},$$
$$\psi(z) = \exp(-2\pi i(z + \bar{z})) \qquad \text{if } F = \mathbb{C},$$

then $\mathscr{S}_0(\mathcal{M})$ is invariant under Fourier Transform. We fix such choices.

The main result of this section is the following.

Theorem 3.2.1. *Suppose π is an irreducible admissible $(\mathfrak{g}, K)$-module.*

1. *There exists a real number s_0, such that the integral $Z(\Phi, s, f)$ in (2.1.1) converges absolutely for $\Re(s) > s_0$ and for all $\Phi \in \mathscr{S}(\mathcal{M})$ and all coefficient f of π.*
2. *The space $I(\pi, s)$ of all $Z(\Phi, s + \frac{1}{2}(n-1), f)$, where $\Phi \in \mathscr{S}_0(\mathcal{M})$ and f any coefficient, is a $\mathbb{C}[s]$-submodule of the space of all meromorphic functions on $\mathbb{C}$.*
3. *There is an Euler factor $L(s, \pi)$ which is a fixed nonzero meromorphic function in s such that $I(\pi, s) = L(s, \pi)\mathbb{C}[s]$. In addition, $Z(\Phi, s, f)/L(s, \pi)$ has an entire continuation for all $\Phi \in \mathscr{S}(\mathcal{M})$.*
4. *There is a meromorphic function $\gamma(s, \pi, \psi)$ so that for all $\Phi \in \mathscr{S}(\mathcal{M})$*
$$Z(\hat{\Phi}, 1 - s + \frac{1}{2}(n-1), \check{f}) = \gamma(s, \pi, \psi) Z(\Phi, s + \frac{1}{2}(n-1), f).$$
If we define the ϵ-factor as in (2.2.2), it is a constant with our choice of ψ above.

The proof of Theorem 3.2.1 is similar to that of Theorem 2.1.2 using parabolic induction. One first has the following definition and results.

Definition 3.2.2. Let B be the standard minimal parabolic (i.e. Borel) subgroup of G of type $(1, \dots, 1)$, and (σ, W) a finite dimensional smooth representation of B trivial on $N(B)$. The *induced representation* $\mathrm{Ind}_B^G \sigma$ of G is defined to be

1. The vector space V of measurable functions $G \to W$ square integrable over K.
2. For any $f \in V$, $b \in B$ and $g \in G$ we have
$$f(bg) = \delta_B(b)^{\frac{1}{2}} \sigma(b) f(g).$$
3. The action of G is by right translation.

Proposition 3.2.3. *The induced representation $\mathrm{Ind}_B^G \sigma$ is admissible.*

Proof. It is [Wal79], Theorem 4.9. □

Theorem 3.2.4 (Casselman's submodule theorem)**.** *Suppose (π, V) be an irreducible $(\mathfrak{g}, K)$-module, then it is isomorphic to a submodule of the associated $(\mathfrak{g}, K)$-module of $\mathrm{Ind}_B^G \sigma$ for some irreducible representation σ of $B/N(B)$.*

Proof. See [Wal79], Theorem 4.18. □

The remaining of the proof is then similar to the nonarchimedean case: first for $\Phi \in \mathscr{S}_0(\mathcal{M})$, one proves a similar statement of Proposition 2.3.3 when $P = B$, and then combine it with Theorem 3.2.4 and the rank 1 case we already know [Tat67] one proves all claims for $\Phi \in \mathscr{S}_0(\mathcal{M})$. Next, using that $\mathscr{S}_0(\mathcal{M})$ is dense in $\mathscr{S}(\mathcal{M})$ and that Fourier transform is continuous on the Schwartz space, one can prove the entire continuation of $Z(\Phi, s, f)/L(s, \pi)$ as in [Jac79], Proposition 4.5 and it is a uniform limit of $Z(\Phi_i, s, f)/L(s, \pi)$ for a sequence of $\Phi_i \in \mathscr{S}_0(\mathcal{M})$ on any vertical strips $s_0 < s_1 < \Re(s) < s_2$. The latter sequence is also uniformly Cauchy on arbitrary vertical strips. Taking limit on both sides of the functional equation one completely proves Theorem 3.2.1.

4. Global Theory

4.1. **The main theorem.** Let F now be a global field, v any place of F, $\mathbf{A} = \mathbf{A}_F$ the ring of Adeles, and $\mathbf{J} = \mathbf{A}_F^\times$ the multiplicative group of Ideles. It is also useful to denote $\mathbf{A}_{\mathrm{fin}}$ the finite adeles (ignoring the infinite part), and $\mathbf{A}_\infty$ the product of F_v when v is infinite. Let $G_{\mathbf{A}} = \mathrm{GL}(n, \mathbf{A})$ and $K = \prod_v K_v$ the standard maximal compact subgroup, where K_v is the one of G_v. We use G_{fin} for $\mathrm{GL}(n, \mathbf{A}_{\mathrm{fin}})$ and similarly G_∞, as well as the subscripts fin or ∞ for other relavent subgroups and Lie algebras. Note $G_{\mathbf{A}}$ can be identified with the restricted direct product of G_v with respect to K_v. We denote $\mathcal{M}_{\mathbf{A}} = \mathrm{Mat}(n, \mathbf{A})$ the space of $n \times n$ matrices, and $\mathscr{S}(\mathcal{M}_{\mathbf{A}})$ the space of Schwartz-Bruhat functions on $\mathcal{M}_{\mathbf{A}}$, the elements of which are linear combinations of functions of the form $\Phi(x) = \prod_v \Phi_v(x_v)$, where $\Phi_v \in \mathscr{S}(\mathcal{M}_v)$ and for almost all v equals the characteristic function of $\mathrm{Mat}(n, R_v)$ (see Remark 4.2.3). $\mathscr{S}(\mathcal{M}_{\mathbf{A}})$ is clearly stable under Fourier transform

$$\hat{\Phi}(x) = \int_{\mathcal{M}_{\mathbf{A}}} \Phi(y)\psi(\mathrm{tr}(yx))\mathrm{d}y,$$

where ψ is a nontrivial additive character of $\mathbf{A}$ trivial on F. If F is a number field, we also denote by $\mathscr{S}_0(\mathcal{M}_{\mathbf{A}})$ the subspace of elements with archimedean component inside $\mathscr{S}_0(\mathcal{M}_\infty)$.

We present the main theorem first and introduce the remaining undefined notions gradually in our sketch of proof.

Theorem 4.1.1. *Suppose $\pi = \otimes_v \pi_v$ is an irreducible automorphic representation of $G_{\mathbf{A}}$ with a central quasicharacter trivial on $F^\times$, $\tilde{\pi}$ its contragredient, $\Phi \in \mathscr{S}(\mathcal{M}_{\mathbf{A}})$, $s \in \mathbb{C}$ and f a coefficient of π.*

(1) *The zeta integral* (2.1.1) *converges absolutely on some right half-plane $\Re(s) > s_0$, and can be analytically continued to the whole $\mathbb{C}$. Moreover, it satisfies the functional equation*

$$Z(\Phi, s + \frac{1}{2}(n-1), f) = Z(\hat{\Phi}, 1 - s + \frac{1}{2}(n-1), \check{f}), \tag{4.1.1}$$

where $\hat{\Phi}$ denotes the Fourier transform with respect to ψ, and $\check{f}(g) = f(g^{-1})$.

(2) *The global L-functions defined by Euler products*

$$L(s, \pi) = \prod_v L(s, \pi_v), \quad L(s, \tilde{\pi}) = \prod_v L(s, \tilde{\pi}_v)$$

as well as the global ϵ-factor

$$\epsilon(s, \pi) = \prod_v \epsilon(s, \pi_v, \psi_v)$$

converge absolutely on some right half-plane, and have analytic continuations to the whole $\mathbb{C}$. Moreover, there is a functional equation

$$L(s, \pi) = \epsilon(s, \pi) L(1 - s, \tilde{\pi}).$$

4.2. **Admissible representations and the global Hecke algebra.** We start by defining the notion of admissible representation for $G_{\mathbf{A}}$, and if F is a number field it is not an actual $G_{\mathbf{A}}$ representation because of the archimedean places.

Definition 4.2.1. An *admissible representation* (π, V) of $G_{\mathbf{A}}$ is a complex vector space V which is simultaneously an admissible representation of G_{fin} and an admissible $(\mathfrak{g}_\infty, K_\infty)$-module, with both actions denoted by π and compatible (i.e. commutative).

It is also convenient to formulate this notion in the language of Hecke algebra.

Definition 4.2.2. Suppose we have a family of vector spaces $\{V_i\}_{i \in I}$, a finite subset $T \subset I$ and each $i \notin T$ a fixed element $e_i \in V_i$. The *restricted tensor product* of $\{V_i\}$ with respect to $\{e_i\}$ is defined as

$$\otimes_{e_i} V_i = \varinjlim_{T \subset S \text{ finite}} (\otimes_{i \in S} V_i) \otimes (\otimes_{i \notin S} e_i).$$

Remark 4.2.3. The global Schwartz functions $\mathscr{S}(\mathcal{M}_{\mathbf{A}})$ is the restricted tensor product of $\mathscr{S}(\mathcal{M}_v)$ with respect to the characteristic functions on $\mathrm{Mat}(n, R_v)$.

Definition 4.2.4. For each nonarchimedean v, let $e_v \in \mathcal{H}(G_v)$ be the characteristic function on K_v, normalized by the volume of K_v. The *global Hecke algebra* $\mathcal{H}(G_{\mathbf{A}})$ is the restricted tensor product $\otimes_{e_v} \mathcal{H}(G_v)$ with the obvious algebra multiplication, where for archimedean places $\mathcal{H}(G_v) = \mathcal{H}(G_v, K_v)$.

There is an analogous definition of admissible representation of $\mathcal{H}(G_{\mathbf{A}})$ in [JL70] using elementary idempotents just as in local cases, and it is readily seen an admissible representation of $G_{\mathbf{A}}$ is equivalent to one of $\mathcal{H}(G_{\mathbf{A}})$ by the following factorization theorem (Theorem 3.3.3 of [Bum97]).

Theorem 4.2.5. *Let (π, V) be an irreducible admissible representation of $G_{\mathbf{A}}$ (resp. $\mathcal{H}(G_{\mathbf{A}})$), then there exists, up to equivalence, for each archimedean place v an irreducible admissible $(\mathfrak{g}_v, K_v)$-module (resp. representation of $\mathcal{H}(G_v, K_v)$), and for each nonarchimedean place v an irreducible admissible representation of G_v (resp. $\mathcal{H}(G_v)$), in both cases denoted by (π_v, V_v), such that for almost all v, V_v contains a nonzero K_v-fixed vector e_v and that V is a restricted tensor product of V_v with respect to e_v and π is the tensor product action $\otimes_v \pi_v$.*

4.3. **Some reduction theory and automorphic representations.** The global part of the theory is essentially just "multiplying" all the local part together. Thus in order to reach a global theory, we need to find a class of factorizable representations abundant enough in which we can always find one object that at each local place gives the correct local Euler factor. This class turns out to be the representations on automorphic forms, which are certain functions on $G_{\mathbf{A}}$. We first need some facts from the reduction theory of $G_{\mathbf{A}}$.

Let T be a maximal F-split torus of G, of which the choice turns out to be irrelavent [GJ72], so we may just choose the standard one being the subgroup of invertible diagonal matrices. Let Z be the center of G, B the standard Borel subgroup, $N = N(B)$ the unipotent radical of B, and G_0 be the kernel of the quasicharacter $g \mapsto |\det g|$. When dealing with their $\mathbf{A}$-points, we can treat all of them as explicit subgroups of $G_{\mathbf{A}}$, which in coordinates is a subspace of $\mathbf{A}^{n^2+1}$ embedded via $(g_{ij}) \mapsto ((g_{ij}), (\det g)^{-1})$. We also use the subscript 0 to denote any intersection with G_0. One first has the fundamental result.

Theorem 4.3.1. *The quotient space $Z_{\mathbf{A}} G_F \backslash G_{\mathbf{A}}$ has finite volume.*

Proof. See Section 14.4 of [Hum80]. □

Definition 4.3.2. With notations above, we denote by $\mathscr{F}(G_F \backslash G_{\mathbf{A}})$ the (measurable) functions on the quotient space $G_F \backslash G_{\mathbf{A}}$, and suppose $\varphi \in \mathscr{F}(G_F \backslash G_{\mathbf{A}})$.

(1) φ is called *cuspidal* if for any proper parabolic subgroup P defined over F, one has

$$\int_{N(P)_F \backslash N(P)_{\mathbf{A}}} \varphi(ug) \mathrm{d}u = 0 \quad \text{for all } g \in G_{\mathbf{A}}.$$

We denote the subspace of cuspidal elements by $\mathscr{F}_0(G_F \backslash G_{\mathbf{A}})$.

(2) Suppose ω is a quasicharacter of $\mathbf{J}$, φ is called an *eigenfunction* of ω if

$$\varphi(ag) = \omega(a)\varphi(g) \quad \text{for all } g \in G_{\mathbf{A}} \text{ and } a \in Z_{\mathbf{A}}.$$

We denote the subspace of ω-eigenfunctions by $\mathscr{F}(G_F \backslash G_{\mathbf{A}}, \omega)$.

As we will demonstrate later, the correct representations of $G_{\mathbf{A}}$ to look at is the ones on automorphic forms, a certain subspace of $\mathscr{F}(G_F \backslash G_{\mathbf{A}})$, in analogue with the classical picture $\Gamma \backslash G_{\mathbb{R},+}$ (Γ a discrete subgroup). Just as in the classical case (a visible one being that of modular forms), we need the functions to have "good" behaviors at "cusps (infinities)" of the quotient space. Thus our hope is to construct a convenient fundamental domain $\mathfrak{s}$ such that $G_{\mathbf{A}} = G_F \mathfrak{s}$. There are several equivalent ways to achieve this, and here we adopt the one in [GJ72]; one can also see [PS79] and [Bum97]. We start with height functions.

Definition 4.3.3. Let T be as above, $t = \operatorname{diag}(t_1, \ldots, t_n) \in T_{\mathbf{A}}$ any element, and $\alpha_i(t) = t_i/t_{i+1}, (1 \le i \le n-1)$ a system of simple roots of T. We define the minimal (resp. maximal) *height function* by

$$\eta(t) = \inf_i |\alpha_i(t)|, \quad \text{resp. } \kappa(t) = \sup_i |\alpha_i(t)|.$$

Definition 4.3.4. Suppose X be a closed subset of $\mathbf{J}$ such that the absolute value is a proper map, C_G a compact subset of $G_{\mathbf{A}}$, C_N a compact subset of $N_{\mathbf{A},0}$, and $\eta_0 > 0$ a real number. A *Siegel domain* $\mathfrak{s} = \mathfrak{s}(X, C_G, C_N, \eta_0)$ is the subset of elements with the form $h = utg$ where $u \in C_N$, $t \in T_{\mathbf{A}}$ with $\alpha_i(t) \in X$ for all i and $\eta(t) > \eta_0$ (such t will be called *semi-bounded*), and $g \in C_G$.

Remark 4.3.5. Note the proper map condition on X is a very loose restriction, for example X could be one of those sets $\{a \in \mathbf{J} \mid |a_v|_v \ge c_v > 0 \text{ and } c_v = 1 \text{ for almost all } v\}$.

Remark 4.3.6. By the semi-boundedness of t, one sees that $\bigcup_t t^{-1} C_N t$ is relatively compact, hence we may also write $h = utg = tg'$ for g' in some fixed compact subset of $G_{\mathbf{A}}$.

The reduction theory then gives the following.

Theorem 4.3.7. *There exists a choice of* X, C_G, C_N *and* η_0 *such that the Siegel domain* $\mathfrak{s}(X, C_G, C_N, \eta_0)$ *is a fundamental domain of left translation by* G_F, *i.e.*

(1) $G_{\mathbf{A}} = G_F\mathfrak{s}$.
(2) *The set* $\{\gamma \in G_F \mid \mathfrak{s} \cap \gamma\mathfrak{s} \neq \varnothing\}$ *is finite.*

Proof. See [God95], where a slightly different decomposition of a Siegel domain is used, but one can manipulate it into the form we have here. □

Now we see the name height function is really justified and we can define growth speed for elements in $\mathscr{F}(G_F\backslash G_{\mathbf{A}})$: since $\mathfrak{s}$ is a union of cosets of $Z_{\mathbf{A}}$, and $Z_{\mathbf{A}}\backslash\mathfrak{s}$ has finite volume by Theorem 4.3.1, one then only needs to look at t in the decomposition of an element in $\mathfrak{s}$, and since such t is semi-bounded, the only "infinity" (modulo $Z_{\mathbf{A}}$) can happen is that $\kappa(t) \to \infty$. More precisely, we have the following definition.

Definition 4.3.8. Suppose $\varphi \in \mathscr{F}(G_F\backslash G_{\mathbf{A}})$.

(1) φ is called *slowly increasing* if for any compact subset C of $G_{\mathbf{A}}$ and any $\eta_0 > 0$, there is $p \geq 1$ and $c > 0$ such that
$$\varphi(tg) = c\kappa(t)^p \quad \text{for all } g \in C \text{ and } t \in T_{\mathbf{A},0}, \eta(t) \geq \eta_0.$$
(2) φ is called *rapidly decreasing* if for any compact subset C of $G_{\mathbf{A}}$, any $\eta_0 > 0$ and any $p \in \mathbb{Z}$ there is $c = c(p) > 0$ such that
$$\varphi(tg) = c\kappa(t)^p \quad \text{for all } g \in C \text{ and } t \in T_{\mathbf{A},0}, \eta(t) \geq \eta_0.$$

We can now define the notions of automorphic form and automorphic representation.

Definition 4.3.9. An *automorphic form* is a complex function φ on $G_F\backslash G_{\mathbf{A}}$ satisfying the following conditions:

(1) φ is $K_{\mathbf{A}}$-finite on the right (hence continuous).
(2) The representation of $\mathcal{H}(G_{\mathbf{A}})$ on the space $\varphi * \mathcal{H}(G_{\mathbf{A}})$ is admissible.
(3) If F is a number field, φ is slowly increasing.

We denote by $\mathscr{A}(G_F\backslash G_{\mathbf{A}})$ the space of automorphic forms and, for each quasicharacter ω of $F^\times\backslash\mathbf{J}$, by $\mathscr{A}(G_F\backslash G_{\mathbf{A}}, \omega)$ the subspace of ω-eigenfunctions.

Definition 4.3.10. An *automorphic representation* of $G_{\mathbf{A}}$ is one that can be realized as a component of $\mathscr{A}(G_F\backslash G_{\mathbf{A}}, \omega)$, where ω is some quasicharacter of $F^\times\backslash\mathbf{J}$.

Remark 4.3.11. The second condition for automorphic forms is in terms of Hecke algebra. We adopt this because it gives directly the property we want: any irreducible automorphic representation is admissible. If we replace it by the condition that φ is $\mathcal{Z}_\infty$-finite where $\mathcal{Z}_\infty$ is the center of $\mathcal{U}(\mathfrak{g}_\infty)$, we get an equivalent definition purely in terms of the group [BJ79].

Definition 4.3.12. A *cusp form* is an automorphic form that is cuspidal. The space of cusp forms is denoted by $\mathscr{A}_0(G_F\backslash G_{\mathbf{A}})$ and ω-eigenspace by $\mathscr{A}_0(G_F\backslash G_{\mathbf{A}}, \omega)$. Similarly we define a *cuspidal representation* to be one realizable as a component of $\mathscr{A}_0(G_F\backslash G_{\mathbf{A}}, \omega)$.

One of the reasons we look at admissible representations at all is that in general not all interesting representations of $G_{\mathbf{A}}$ can be made unitary (e.g. Eisenstein series according to [Bum97]); however, for a cuspidal representation, unitarity is always possible up to a twisting.

Lemma 4.3.13. *Suppose* $\varphi \in \mathscr{F}_0(G_F\backslash G_{\mathbf{A}}, \omega)$ *for some quasicharacter (resp. character)* ω *of* $F^\times\backslash\mathbf{J}$.

(1) *If* F *is a number field, and* φ *is slowly increasing (resp. square integrable modulo* $Z_{\mathbf{A}}G_F$*), then* $\varphi * f$ *is rapidly decreasing for any* $f \in \mathcal{H}(G_{\mathbf{A}})$.
(2) *If* F *is a function field, and* φ *is* K*-finite on the right, then* φ *is compactly supported modulo* $Z_{\mathbf{A}}G_F$.

Proof. See [GJ72], Lemmas 10.8 and 10.9. □

Suppose ω is now a (unitary) character, let $L^2(G_F\backslash G_{\mathbf{A}}, \omega)$ be the Hilbert space of ω-eigenfunctions that are square-integrable on $Z_{\mathbf{A}}G_F\backslash G_{\mathbf{A}}$ (note it is well defined since ω is a character), and $L_0^2(G_F\backslash G_{\mathbf{A}}, \omega)$ the closed subspace of cuspidal elements. $G_{\mathbf{A}}$ has a natural (actual!) representation on such spaces by right translation. Lemma 4.3.13 allows us to prove the following.

Proposition 4.3.14. *Suppose ω is a character of $F^\times\backslash\mathbf{J}$, then $L_0^2(G_F\backslash G_{\mathbf{A}}, \omega)$ decomposes into Hilbert space direct sum of unitary irreducible admissible representations of $G_{\mathbf{A}}$. Moreover, $\mathscr{A}_0(G_F\backslash G_{\mathbf{A}}, \omega)$ is the dense subspace of K-finite and $\mathcal{Z}_\infty$-finite (when F is a number field) vectors, and decomposes into algebraic direct sum of preunitary (contained as a dense subspace of a unitary representation) irreducible admissible representations of $\mathcal{H}(G_{\mathbf{A}})$.*

Proof (sketch). By definition, any cusp form φ can be written as $\varphi * f$ for some $f \in \mathcal{H}(G_{\mathbf{A}})$, hence by Lemma 4.3.13 it is always rapidly decreasing hence belongs to $L^2(G_F\backslash G_{\mathbf{A}})$. For function fields one can prove the representation on L_0^2 is admissible as in Corollary 10.10 of [GJ72], while for number fields it involves a more subtle decomposition of L_0^2 into eigenspaces of $\mathcal{Z}_\infty$. For more details one can see [GJ72]. □

Corollary 4.3.15. *Suppose ω is a quasicharacter of $F^\times\backslash\mathbf{J}$. The representation of $\mathcal{H}(G_{\mathbf{A}})$ on $\mathscr{A}_0(G_F\backslash G_{\mathbf{A}}, \omega)$ decomposes into a direct sum of irreducible admissible representations.*

Proof. By twisting the representation on $\mathscr{A}_0(G_F\backslash G_{\mathbf{A}}, \omega)$ with $|\omega(\det(\cdot))|^{-1/n}$, we get a unitary one hence have a desired decomposition by Proposition 4.3.14. Twisting it back we get the result. □

Since the contragredient to the representation on $\mathscr{A}_0(G_F\backslash G_{\mathbf{A}}, \omega)$ is just the one on $\mathscr{A}_0(G_F\backslash G_{\mathbf{A}}, \omega^{-1})$, and the coefficient f determined by $\varphi \in \mathscr{A}_0(G_F\backslash G_{\mathbf{A}}, \omega)$ and $\tilde{\varphi} \in \mathscr{A}_0(G_F\backslash G_{\mathbf{A}}, \omega^{-1})$ is the integral

$$f(g) = \int_{Z_{\mathbf{A}}G_F\backslash G_{\mathbf{A}}} \varphi(hg)\tilde{\varphi}(h)\mathrm{d}h. \tag{4.3.1}$$

In view of the zeta integral (2.1.1), one may just assume ω is always a character.

Finally, the following result (from the supplement by Langlands to [BJ79]) , analogous to Theorems 2.3.9 and 3.2.4, is needed for parabolic induction.

Theorem 4.3.16. *Every irreducible automorphic representation π is a component of $\mathrm{Ind}_{P_{\mathbf{A}}}^{G_{\mathbf{A}}}\sigma = \otimes_v \mathrm{Ind}_{P_v}^{G_v}\sigma_v$ for some standard parabolic subgroup P of type $(n_1, \dots, n_r)$ and a cuspidal representation $\sigma = \boxtimes_{1\le i\le n_r}\sigma_i$ of $P_{\mathbf{A}}/N(P)_{\mathbf{A}}$.*

4.4. **The final blow.** The proof of Theorem 4.1.1 in the cuspidal case follows a similar path as in Tate's thesis [Tat67]. In this subsection all omitted proofs can be found in Sections 11–13 of [GJ72].

First of all, let $\{\lambda_0, \lambda_1\}$ be a smooth partition of unity on $\mathbb{R}_+^\times$, satisfying $\lambda_0(t) = \lambda_1(t^{-1})$ and that λ_0 is supported away from 0 and equals 1 for all large t. Suppose ω is a character of $F^\times\backslash\mathbf{J}$, $s \in \mathbb{C}$, and V a finite dimensional vector space (over F or $\mathbf{A}$). For $i = 0, 1$, we define two linear functionals on $\mathscr{S}(V_{\mathbf{A}})$:

$$\langle\theta^i(s,\omega), \Phi\rangle = \int_{F^\times\backslash\mathbf{J}} \sum_{\xi\in V_{\mathbf{A}}-\{0\}} \Phi(a\xi)|a|^s\omega(a)\lambda_i(|a|)\mathrm{d}a,$$
$$\langle\check{\theta}^i(s,\omega), \Phi\rangle = \langle\theta^i(s,\omega), \hat{\Phi}\rangle.$$

Lemma 4.4.1. *With the notations above, the integral $\langle\theta^0(s,\omega), \Phi\rangle$ converges for all s and uniformly for s in any compact subset of $\mathbb{C}$. Moreover, the function $(g, s) \mapsto \langle\theta^0(s,\omega), \Phi.g\rangle$ on $G_{\mathbf{A}} \times \mathbb{C}$ is holomorphic in s and slowly increasing in g, where $(\Phi.g)(x) = \Phi(gx)$. The same holds for $\theta^1(s,\omega)$ as well when $\Re(s) > \dim_F(V_F)$.*

Proposition 4.4.2 (Poisson summation formula)**.** *Suppose $V_{\mathbf{A}}/V_F$ has measure one and $\Phi \in \mathscr{S}(V_{\mathbf{A}})$, we have*

$$\sum_{\xi\in V_F} \Phi(\xi) = \sum_{\xi\in V_F} \hat{\Phi}(\xi).$$

Proof. This follows from the general result for locally compact abelian groups, and the fact that F is discrete and cocompact in $\mathbf{A}$ and the quotient $\mathbf{A}/F$ is the Pontryagin dual to F [Tat67]. □

Suppose now $V_{\mathbf{A}} = \mathcal{M}_{\mathbf{A}}$, for each $0 \le r \le n$ and $i = 0, 1$, we also introduce the following functionals:

$$\langle \theta_r^i(s,\omega), \Phi \rangle = \int_{F^\times \backslash \mathbf{J}} \sum_{\operatorname{rank} \xi = r} \Phi(a\xi)|a|^s \omega(a)\lambda_i(|a|)\mathrm{d}a.$$

Note that $\theta^i = \sum_{r=1}^n \theta_r^i$ and the inner summation in θ_0^i is just $\Phi(0)$, so θ_0^i is also defined for any $V_{\mathbf{A}}$. With the help of Proposition 4.4.2, we can relate θ^0 with θ^1 through a basic calculation.

Corollary 4.4.3. *Suppose ω is a character of $F^\times \backslash \mathbf{J}$ and $Re(s) > n^2$, we have*

$$\theta^1(s,\omega) = \check{\theta}^0(n^2 - s, \omega^{-1}) + \check{\theta}_0^0(n^2 - s, \omega^{-1}) - \theta_0^1(s,\omega),$$

as well as

$$\theta_n^1(s,\omega) = \check{\theta}_n^0(n^2 - s, \omega^{-1}) + \sum_{r=1}^{n-1} \check{\theta}_r^0(n^2 - s, \omega^{-1}) - \sum_{r=1}^{n-1} \theta_r^1(s,\omega) + \check{\theta}_0^0(n^2 - s, \omega^{-1}) - \theta_0^1(s,\omega). \tag{4.4.1}$$

Proof of Theorem 4.1.1. Suppose the matrix coefficient f is the one specified in (4.3.1) we can formally rewrite (2.1.1) into

$$Z(\Phi, s, f) = \int_{G_{\mathbf{A}} \times (Z_{\mathbf{A}} G_F \backslash G_{\mathbf{A}})} \Phi(g)\varphi(hg)\tilde{\varphi}(h)|\det g|^s \mathrm{d}g\mathrm{d}h, \tag{4.4.2}$$

the left-hand side being an iterated integral and the right-hand side a double integral. To prove (4.4.2), one has to study the following integral:

$$Z(\Phi, s, \varphi) = \int_{G_{\mathbf{A}}} \Phi(g)|\det g|^s \varphi(g)\mathrm{d}g.$$

This integral is provably convergent for $\Re(s) > n$. If we choose a finite set of representatives $g_i, 0 \le i \le l$ so that $G_{\mathbf{A}} = \bigsqcup_i Z_{\mathbf{A}} G_0 g_i$ (disjoint union of cosets) and let $G' = \bigsqcup_i G_0 g_i$ and $G'' = G'^{-1} = \bigsqcup_i g_i^{-1} G_0$. Note G_0 is normal in $G_{\mathbf{A}}$ so G' and G'' are both unions of left and right cosets of G_0. Then a bunch of calculation (with some argument about convergence and the help of (4.4.1)) will show that when $\Re(s) > n$

$$\begin{aligned} Z(\Phi, s, \varphi) = &\int_{G_F \backslash G'} \varphi(g)|\det g|^s \langle \theta_n^0(ns, \omega), g.\Phi \rangle \mathrm{d}g + \int_{G_F \backslash G'} \varphi(g)|\det g|^{s-n} \langle \theta_n^0(n^2 - ns, \omega^{-1}), \hat{\Phi}.g^{-1} \rangle \mathrm{d}g \\ &+ \sum_{r=0}^{n-1} \left[\int_{G_F \backslash G'} \varphi(g)|\det g|^{s-n} \langle \theta_r^0(n^2 - ns, \omega^{-1}), \hat{\Phi}.g^{-1} \rangle \mathrm{d}g - \int_{G_F \backslash G'} \varphi(g)|\det g|^s \langle \theta_r^1(ns, \omega), g.\Phi \rangle \mathrm{d}g \right], \end{aligned}$$

where $g.\Phi$ means the right translation by g. The terms in the two summation parts are provably zero when φ is a cusp form. Hence, by a simple change of variable in the second term, we get when $\Re(s) > n$

$$Z(\Phi, s, \varphi) = \int_{G_F \backslash G'} \varphi(g)|\det g|^s \langle \theta_n^0(ns, \omega), g.\Phi \rangle \mathrm{d}g + \int_{G''/G_F} \check{\varphi}(g)|\det g|^{n-s} \langle \theta_n^0(n^2 - ns, \omega^{-1}), \hat{\Phi}.g \rangle \mathrm{d}g.$$

The right-hand side exists for all s and hence by Fubini-Tonelli, we establish the absolute convergence of the right-hand side of (4.4.2) hence also the equality. A little bit more calculation gives the symmetric expression for (2.1.1) we desired, valid for $\Re(s) > n$:

$$\begin{aligned} Z(\Phi, s, f) = &\int_{(G''/G_F) \times (G_F \backslash G')} \check{\tilde{\varphi}}(h)\varphi(g)|\det gh|^s \mathrm{d}h\mathrm{d}g \langle \theta_n^0(ns, \omega), g.\Phi.h \rangle \\ &+ \int_{(G_F \backslash G') \times (G''/G_F)} \tilde{\varphi}(h)\check{\varphi}(g)|\det gh|^{n-s} \mathrm{d}h\mathrm{d}g \langle \theta_n^0(n^2 - ns, \omega^{-1}), h.\hat{\Phi}.g \rangle. \end{aligned} \tag{4.4.3}$$

We also have a similar one for $Z(\hat{\Phi}, s, \check{f})$ using change of variables. (4.4.3) allows us to analytically continue $Z(\Phi, s, f)$ to all s and its symmetry gives the functional equation (4.1.1). This concludes the proof of the first claim of Theorem 4.1.1 for cuspidal representations.

To prove the second claim, one needs to connect the global zeta integral to the local ones, specifically the ones that generate all local L-factors. According to Theorem 4.2.5, for almost all places v, π_v is spherical, hence its L-factor is generated by Φ_v being the characteristic function of $\operatorname{Mat}(n, R_v)$. Thus it is easily shown there is a finite set of indices J_v such that for almost all v J_v contains only one element, and that for each $j \in J = \prod_v J_v$ there is a $\Phi_j = \prod_v \Phi_{j_v} \in \mathscr{S}(\mathcal{M}_{\mathbf{A}})$ and a coefficient $f_j = \prod_v f_{j_v}$, so that

$$\sum_{j_v \in J_v} Z(\Phi_{j_v}, s + \frac{1}{2}(n-1), f_{j_v}) = L(s, \pi_v),$$

and hence

$$\sum_{j \in J} Z(\Phi_j, s + \frac{1}{2}(n-1), f_j) = \prod_v L(s, \pi_v) = L(s, \pi).$$

We also have a similar result on the contragredient side. The second claim is then obvious, and Theorem 4.1.1 is completely proved for cuspidal representations.

To finally prove Theorem 4.1.1 for automorphic representations, one uses Theorem 4.3.16 and the familiar argument of parabolic induction, during which we need to note that, for almost all v, $\mathrm{Ind}_{P_v}^{G_v} \sigma_v$ has exactly one component π_v that is spherical, so we have $L(s, \pi_v) = \prod_i L(s, \sigma_{v,i})$ by Propositions 2.3.3 and 2.5.2. One may also need to utilize the denseness of $\mathscr{S}_0(\mathcal{M}_{\mathbf{A}})$ in $\mathscr{S}(\mathcal{M}_{\mathbf{A}})$ if F is a number field. □

4.5. **Global Example.** In this last subsection we provide some connection between the classical L-functions of cusp forms of $\mathrm{GL}(2, \mathbb{R})_+$ and the standard L-functions of $\mathrm{GL}(2, \mathbf{A}_{\mathbb{Q}})$. However, we will be very brief concerning the computation details. Nice references are [Gel75] and [GS88], as well as [Bum97].

Let $\mathbb{H}_+$ be the upper-half complex plane, $q = \exp(2\pi i z)$, $\Gamma_0 = \mathrm{SL}(2, \mathbb{Z})$, and f be a classical cusp form of weight k on $\Gamma_0 \backslash \mathbb{H}_+$ that is also an eigenfunction for all Hecke operators. Then we know f has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_n q^n,$$

and a_n are multiplicative. In this case, the L-function $L(f, s) = \sum_n a_n n^{-s}$ can also be viewed as the Mellin transform

$$(2\pi)^{-s} \Gamma(s) L(f, s) = \int_{\mathbb{R}_+^\times} f(iy) y^s \mathrm{d}^\times y.$$

Assume further $a_1 = 1$, we have the Euler product expression

$$L(f, s) = \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

Moreover, the Mellin transform may also be seen as a Hecke type zeta integral over $\mathrm{GL}(2) \times \mathrm{GL}(1)$ which ultimately connects to our standard zeta integrals through the Whittaker models.

On the other hand, using isomorphisms $\mathbb{H}_+ \cong \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$ and, according to strong approximation, $Z_{\mathbf{A}} G_{\mathbb{Q}} \backslash G_{\mathbf{A}} / K_{\mathbf{A}} \cong \Gamma_0 \backslash \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$, one can translate the classical cusp form f into an adelic cusp form φ_f via

$$\varphi_f(g) = (c_\infty i + d_\infty)^{-k} f(g_\infty(i)), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The representation π_f generated by right translations of φ_f by $G_{\mathbf{A}}$ is then an irreducible cuspidal representation. For almost all (in this level 1 example, actually for all) p, the local component is an irreducible unramified principal series $\mathcal{B}_p(\chi_{p,1}, \chi_{p,2})$. One can check $\chi_{p,1}$ and $\chi_{p,2}$ satisfy (and are determined by) the relations

$$a_p = p^{\frac{1}{2}(k-1)}(\chi_{p,1}(p) + \chi_{p,2}(p)),$$
$$\chi_{p,1}\chi_{p,2} \equiv 1.$$

The global standard L-function at finite places attached to π_f is then

$$L(\pi_f, s)_{\mathrm{fin}} = \prod_p (1 - a_p p^{-s-\frac{1}{2}(k-1)} + p^{-2s})^{-1}.$$

A change of variable gives

$$L(f, s) = L(\pi_f, s - \frac{1}{2}(k-1))_{\mathrm{fin}}.$$

One can also add the archimedean parts which are $(2\pi)^{-s}\Gamma(s)$ and resp. change s to $s-\frac{1}{2}(k-1)$, as well as the ϵ-factor (in this case $(-1)^{k/2}$) to complete the story of functional equations on both sides.

5. Acknowledgements

I would like to thank Professor Ngô Bao Châu for suggesting this topic and providing insightful guidance during many discussions, Minh-Tam Trinh for some helpful discussions on the archimedean case, and Drew Moore for his help in providing references for global examples. I would also like to thank Mathilde Gerbelli-Gauthier and Eric Stubley for organizing a student local Langlands seminar in the summer of 2017 where I had a chance to access more concrete computations.

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